Many-valued judgment aggregation: characterizing the possibility/impossibility boundary[*]

Conal Duddy[†]and Ashley Piggins[‡]

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Abstract

A general model of judgment aggregation is presented in which judgments on propositions are not binary but come in degrees. The primitives of the model are a set of propositions, an entailment relation, and a "triangular norm" which establishes a lower bound on the degree to which a proposition is true whenever it is entailed by a set of propositions. Under standard assumptions, we identify a necessary and sufficient condition for the collective judgments to be both deductively closed and free from veto power. This condition says that the triangular norm used to establish the lower bound must contain a zero divisor.

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[†] Department of Philosophy, Logic and Scientific Method, London School of Economics, U.K. and J.E. Cairnes School of Business and Economics, National University of Ireland Galway, Ireland. Email: conal.duddy@gmail.com

[‡] J.E. Cairnes School of Business and Economics, National University of Ireland Galway, Ireland. Email: ashley.piggins@nuigalway.ie. Tel.: +353 91 492300; fax: +353 91 524130.

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1 Introduction

A large literature on judgment aggregation now exists, motivated by List and Pettit's [14] initial contribution. List and Polak [15] is a survey to which we refer the interested reader. The literature is concerned with aggregating profiles of individual judgment sets into a collective judgment set. A judgment set is a subset of a given "agenda". The agenda is simply a set of propositions upon which a collective judgment is sought. An individual's judgment set contains only those propositions in the agenda that the individual believes to be true. Of particular interest in the literature are agendas that contain logically interconnected propositions. Take for example an agenda of the form $\{p, q, p \wedge q\}$ where p, q and $p \wedge q$ are standard propositions from classical propositional logic. In this case, logical consistency requires that a judgment set must contain $p \wedge q$ if it contains both p and q. Another that contains p but excludes q must exclude $p \wedge q$. A third must exclude $p \wedge q$ if it contains q but not p.[1]

Suppose that these three judgments sets are held by individuals 1, 2 and 3 respectively. Using the numbers 1 for "true" and 0 for "false", we can represent these judgments in the following table.

	p	q	$p \wedge q$
Individual 1	1	1	1
Individual 2	1	0	0
Individual 3	0	1	0
Majority judgment	**1**	**1**	**0**

Table 1: A discursive paradox.

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[1] In our model, if an individual believes that these three propositions are false, then their judgment set is empty. This differs from the standard representation in which this individual's judgment set would be non-empty and contain $\neg p$, $\neg q$ and $\neg(p \wedge q)$. The difference follows from the fact that we do not require the agenda to be a set of proposition/negation pairs. See List and Polak [15] for details.

As is well-known, applying majority voting to aggregate these three judgment sets produces a collective judgment set containing p and q but excluding $p \wedge q$. This is a logically inconsistent set of beliefs. The theory of judgment aggregation has been developed in response to observations such as this.[2]

The logical inconsistency identified in this example is more precisely a failure of deductive closure.[3] Classical propositional logic tells us that the set of propositions $\{p, q\}$ entails the proposition $p \wedge q$. Given this entailment, deductive closure says that if the collective judgment is that p is true (i.e. the collective judgment set contains p) and also that q is true, then the collective judgment must be that $p \wedge q$ is true. Although it seems attractive to determine collective judgments through majority voting when these majorities exist (like in Table 1), the example illustrates that this can produce judgments that fail to be deductively closed.

In this example, like much of the literature, judgments on propositions are binary. This means that for each proposition p in the agenda, individuals (and the collective) must judge that either p is true or that p is false (i.e. an individual judgment set either contains p or excludes p). This paper relaxes this assumption and allows propositions to be judged true to a degree. As is common, we will assume that these degrees are elements of $[0, 1]$. Models that admit degrees of truth are called "many-valued" models.[4]

Allowing propositions to be judged true to a degree is a natural and less restrictive assumption to make about judgments, and one that allows us to consider a richer family of aggregation rules. To take an example from Dietrich [4], imagine that p is the sentence "The birth rate is too low to guarantee long-term economic stability". Clearly it is too demanding to insist that judgments on p be binary. An individual might believe that p is more true than false, and yet feel uneasy about saying that it is definitely true. Many-valued models are one possible way of reflecting this lack of confidence (at both the individual level and the collective level).

As noted above, new aggregation rules are possible in a many-valued setting. To illustrate, consider the following table.

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[2] Although this example appears similar to the standard majority voting paradox, the latter is actually a special case of one of these so-called "discursive dilemmas". See Dietrich and List [5].

[3] See Dietrich and List [6]. Stated informally, deductive closure says that you accept the logical consequences of your beliefs.

[4] See Priest [20] for details.

	p	q	$p \wedge q$
Individual 1	1	1	1
Individual 2	1	0	0
Individual 3	0	1	0
Average judgment	$\mathbf{\frac{2}{3}}$	$\mathbf{\frac{2}{3}}$	$\mathbf{\frac{1}{3}}$

Table 2: Taking averages.

In this example, individual judgments are identical to those in our earlier example (in the example, individuals are perfectly confident in their judgments). However, the collective judgment on each proposition is an average of the individual judgments on this proposition. This highlights a potential advantage of many-valued models. Many-valued models allow us to "smooth" the aggregation of individual judgments, potentially making the problem of judgment aggregation easier to solve.[5] However, this begs a fundamental question: how should we formulate the requirement of deductive closure in a many-valued setting? This is important because we would like to know whether or not the collective judgments in Table 2 are, in fact, deductively closed. If so, then this might suggest that taking averages is a perfectly sound method of aggregation.

In this paper we propose a definition of deductive closure in a many-valued setting. This definition of deductive closure makes use of a mathematical function called a triangular norm.[6] We are neutral as to the exact form this function takes. There are infinitely many possible functional forms. Importantly, whichever norm we happen to use, our definition of deductive closure collapses into the standard one when judgments on propositions are required to be binary.

In order to motivate our central theorem, we prove first of all that under one particular triangular norm the collective judgments produced by averaging will in fact, for every agenda and for every profile of individual judgments, be deductively closed (this is Theorem 1).

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[5] The use of many-valued models in social choice theory is discussed in Barrett and Salles [2] and Salles [22].

[6] A comprehensive reference on triangular norms and their applications is Klement, Mesiar and Pap [13]. An application of these concepts to conventional preference aggregation can be found in Duddy, Perote-Peña and Piggins [11].

Averaging is a natural method of aggregation and one that satisfies certain normative properties common in the social choice literature. Averaging satisfies the independence condition (this says that the collective judgment on proposition p is determined by individual judgments on p). It also satisfies the unanimity condition (if everyone holds identical judgments then these are also the collective judgments). Finally, taking averages means that no individual is a veto-dictator (it is impossible under the averaging rule for one individual to always ensure that a proposition is judged false simply by believing it to be false herself).

Our central theorem (Theorem 4) says the following. We can partition the set of triangular norms into two parts. One part contains all and only those triangular norms under which, no matter what the agenda, a method of aggregation with those three properties (independence, unanimity and freedom from veto-dictatorship) exists that will always produce deductively closed collective judgments. Included in this part of the partition is, of course, the triangular norm under which averaging always produces deductively closed collective judgments.

It follows from this that the other part of the partition contains all and only those triangular norms under which, for some agenda, no aggregation rule exists that satisfies our three normative properties (and can also be relied upon to produce deductively closed collective judgments). In other words, for every norm in this part of the partition, there exists an agenda at which deductive closure at the collective level requires that we (1) either make at least one individual a veto-dictator, or (2) use an aggregation rule that violates unanimity or independence. Furthermore, we identify a set of agendas (what we call "non-trivially, totally blocked agendas") at which these failures of existence occur.[7] Every agenda in this set is one at which, for every triangular norm in this part of the partition, no aggregation rule can exist that satisfies our three properties and also produce deductively closed collective judgments.[8]

In the language of social choice theory, Theorem 4 characterizes the possibility/impossibility boundary in our model. Normatively defensible judgment aggregation is possible at one side of this boundary, but it is impossible at the other. Theorem 4 describes how this boundary is drawn (i.e. it describes

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[7] As we explain later, the agenda $\{p, q, p \wedge q\}$ is non-trivially, totally blocked.

[8] Note that we do not characterize all agendas for which this failure of existence occurs, but we identify a set at which it does.

how the partition of triangular norms is determined).

The recent literature on judgment aggregation contains a small number of papers on many-valued aggregation.[9] Pauly and van Hees [19] and van Hees [12] are two important contributions in which impossibility results are obtained in a specific logical framework, and under a richness assumption about the agenda. Our approach differs from theirs in that we do not work with any particular formal language (as we explain in the next section, we adopt what Dietrich [4] calls "general logic"). Using the framework of general logic, together with a generalization of the standard deductive closure condition, enables us to completely characterize the possibility/impossibility boundary. The fact that possibility can arise at all in a many-valued setting is surprising, and goes against the grain of this earlier literature.

We conclude (in Section 3.2) by giving a non-formal interpretation of our result. We also explain what happens in the model when judgments on propositions are required to be binary.

2 Model

In this section we present our formal model.

2.1 Agenda

Following Dietrich [4] we represent a logic (or a "formal language") abstractly as an ordered pair $(L, \vDash)$ where L is a non-empty set of formal expressions (called **propositions**), and $\vDash$ $(\subseteq \mathcal{P}(L) \times L)$ is a relation between sets $A \subseteq L$ and propositions $p \in L$.[10] L tells us what propositions can be formed in the logic, and $\vDash$ tells us how they are logically interrelated. The relation $\vDash$ is called an **entailment relation**. The set of propositions and the entailment relation are primitive concepts in our model.

One advantage of having these concepts as primitives is that we do not have to commit ourselves to an explicit formal language: the results we obtain apply to a range of logics, not just classical propositional logic.[11] In

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[9] A many-valued interpretation can be given to results in Dietrich [3]. Dokow and Holzman [10] consider non-binary aggregation in an abstract model that is similar to Wilson [24] and Rubinstein and Fishburn [21]. Many-valuedness is also considered explicitly in Dietrich and List [7] who derive a counterpart to our Lemma 3.

[10] $\mathcal{P}(L)$ is the set of non-empty subsets of L.

[11] This advantage is highlighted by Dietrich [4].

our model, any logic that satisfies two conditions is one to which our theorem will apply. These two conditions say the following: for all $A, B \subseteq L$ and all $p \in L$, (1*) if $p \in A$ then $A \vDash p$, and (2*) if $A \vDash b$ for all $b \in B$, and $B \vDash p$, then $A \vDash p$. Condition (1*) is a stronger version of Dietrich's [4] "self-entailment" condition. Condition (2*) is what Dietrich calls (quite naturally) "transitivity".

Our original example was based on the idea that $\{p, q\}$ entails $p \wedge q$ in classical propositional logic. We write $\{p, q\} \vDash p \wedge q$ to denote this. Of course, this logic also tells us that $\{p, q\} \vDash p \vee q$, $\{p, p \rightarrow q\} \vDash q$, $\{p\} \vDash p$, and so on.

The **agenda** is simply a finite, non-empty subset of L. We denote the agenda by X.

Our main theorem (Theorem 4) refers to the possibility of an agenda being "non-trivially, totally blocked". To explain this, let us distinguish between two kinds of entailment. Consider again the example of an agenda $\{p, q, p \wedge q\}$. We say that $p \wedge q$ *directly entails* p because $\{p \wedge q\} \vDash p$. The reverse is not true; p does not directly entail $p \wedge q$. We do however say that p *conditionally entails* $p \wedge q$, because $\{p, q\} \vDash p \wedge q$. In this case the entailment is conditional on q. Note that we do not say that p conditionally entails q, despite the fact that $\{p, q\} \vDash q$. This is because of the redundancy of p here. More generally, for any given agenda X and all $p, q \in X$, we say that p conditionally entails q if there exists a non-empty subset $A \subseteq X$ such that $(A \cup \{p\}) \vDash q$ while $A \nvDash q$. Let us write $p \rhd q$ if p either directly or conditionally entails q.

We say that an agenda is **totally blocked** if the relation $\rhd$ generates a cycle that consists of all of the propositions in X.[12] And we say that the agenda is **non-trivially**, **totally blocked** if it is totally blocked and there is at least one instance of conditional entailment. The agenda $\{p, q, p \wedge q\}$ gives us an example of an agenda that is non-trivially, totally blocked since we have the cycle

$$p \rhd p \wedge q \rhd q \rhd p \wedge q \rhd p$$

and we know that both p and q conditionally entail $p \wedge q$.

Another example of a non-trivially, totally blocked agenda is the following "preference agenda".[13] Let a, b and c be labels for three alternatives and let the agenda be the set of six propositions $\{aRb, aRc, bRc, bRa, cRa, cRb\}$. The

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[12] This is similar in spirit, although not identical to, the definition of total blockedness in Nehring and Puppe [17, 18]. That condition has been central to recent work on abstract aggregation. See Nehring and Puppe [18] and Dokow and Holzman [8, 9].

[13] See Dietrich and List [5] for details.

proposition aRb is taken to mean "a is at least as good as b", and each of the others is interpreted in the same way. Naturally, by the transitivity of "at least as good as", we have $\{aRb, bRc\} \vDash aRc$ and $\{bRa, aRc\} \vDash bRc$ and $\{bRc, cRa\} \vDash bRa$ and so forth.

This agenda corresponds to the problem of aggregating preferences, which is the classic problem of social choice theory.[14] This preference agenda is non-trivially, totally blocked since we have the cycle

$$aRb \rhd aRc \rhd bRc \rhd bRa \rhd cRa \rhd cRb \rhd aRb$$

and aRb conditionally entails aRc (via bRc).

Let us fix an agenda X for the time being.

2.2 Deductive closure

A judgment is represented by a function from X to $[0, 1]$. A judgment may or may not be deductively closed.

Without loss of generality, return to our original example where $\{p, q\} \vDash p \wedge q$. Let $f : X \to [0, 1]$ represent a judgment. A natural idea is that since $\{p, q\} \vDash p \wedge q$ then $f(p)$ and $f(q)$ ought to determine a **lower bound** for $f(p \wedge q)$. More precisely, we can think of this lower bound as being determined by a function $T : [0, 1]^2 \to [0, 1]$ where $T(f(p), f(q))$ is the lower bound on $f(p \wedge q)$, i.e. $T(f(p), f(q)) \leq f(p \wedge q)$.

What properties are natural to impose on T? It seems reasonable that $T(f(p), f(q)) = T(f(q), f(p))$. This is a commutativity property. We would also expect that $T(f(p), 1) = f(p)$. This property says that the number 1 acts as a neutral element. As we will see later, this condition ensures consistency with the binary case.[15] One implication of this is that if $f(p) = 1$ and $f(q) = 1$ then $f(p \wedge q) = 1$, again a natural requirement. We would also expect that $T(f(p), f^1(q)) \geq T(f(p), f^2(q))$ if $f^1(q) \geq f^2(q)$. In other words, we would expect the lower bound on $f(p \wedge q)$ to be weakly increasing whenever the judgment on q rises (holding $f(p)$ constant). This is a monotonicity property.

Finally, we need to be able to deal with cases in which $A \vDash p$ and A contains more than two propositions. To do this, let us suppose that

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[14] Of course, Arrow [1] is the central reference.

[15] This requires that $T(0, 1) = 0$ and $T(1, 1) = 1$. So $T(a, b) = \frac{a+b}{2}$ would violate this requirement, for instance.

$T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in [0,1]$. This is an associativity property. We can exploit this associativity condition to extend each t-norm in the following way. For all $n > 2$ and every $(x_1, ..., x_n) \in [0,1]^n$, $T(x_1, ..., x_n) = T(T(x_1, ..., x_{n-1}), x_n)$. And, for every $x \in [0,1]$, $T(x) = x$.

T functions that satisfy these properties are called **triangular-norms** (or, simply, t-norms).[16] Here are some examples. The minimum t-norm is defined as $T_M(x, y) = \min(x, y)$. The product t-norm is defined as $T_P(x, y) = x \times y$. The Łukasiewicz t-norm is defined as $T_L(x, y) = \max(x + y - 1, 0)$.

Given a judgment f and a set of propositions $A \in \mathcal{P}(X)$, we write f_A to denote the $|A|$-tuple $(f(a_1), \ldots, f(a_{|A|}))$ where $a_1, \ldots, a_{|A|}$ are the propositions in A in some arbitrary order.

We now present our definition of deductive closure. Fix a triangular norm T. For all non-empty sets of propositions A and for every proposition $p \in X$,

$$A \vDash p \text{ implies that } T(f_A) \leq f(p). \tag{1}$$

A judgment f is deductively closed if and only if it satisfies (1).

One advantage of this formulation is that it collapses into the standard one whenever judgments on propositions are required to be binary. Suppose that it is the case that for every proposition p either p is judged to be true or p is judged to be false. Then, if every proposition in A is judged to be true, it follows that $T(f_A) = 1$ and so, according to (1), p must also be judged to be true.[17] However, if at least one proposition in A is judged to be false then $T(f_A) = 0$ and so p is either true or false.

Returning to Table 2, we can now see that under the norms T_M and T_P, the collective judgments (obtained from averaging) are not deductively closed. Letting f^* denote the collective judgment, we have

$$T_M(f^*(p), f^*(q)) > f^*(p \wedge q) \tag{2}$$

and

$$T_P(f^*(p), f^*(q)) > f^*(p \wedge q). \tag{3}$$

Both (2) and (3) contradict (1). However, using the norm T_L, we have $T_L(f^*(p), f^*(q)) \leq f^*(p \wedge q)$. This does not contradict (1).

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[16] Again, Klement, Mesiar and Pap [13] is an excellent reference.

[17] Condition (1) respects the idea that you believe propositions to be true if they are logical consequences of other things you believe to be true. This is the intuition behind deductive closure.

It turns out that there is an intimate connection between one particular property of T_L and the three normative properties of the averaging rule. This key property of T_L is that is has a “zero divisor”. A t-norm T has a **zero divisor** if there exist x, y in the open interval $(0, 1)$ such that $T(x, y) = 0$. Then x and y are called zero divisors.

Clearly neither T_M nor T_P has a zero divisor. However, T_L has a zero divisor since $T_L(\frac{1}{2}, \frac{1}{2}) = 0$. In fact, since $T_L(x, 1 - x) = 0$ for all $x \in (0, 1)$, then every number in $(0, 1)$ is a zero divisor of T_L.

Let us now fix an arbitrary t-norm T and so set the corresponding notion of deductive closure.

2.3 Aggregation

Let V be the set of all judgments that are deductively closed. Let N be a finite set of natural numbers $\{1, \ldots, n\}$, $n \geq 2$. These numbers represent the individuals in society.

An aggregation rule is a function $\Phi : V^n \to V$. Let ϕ denote $\Phi(v_1, \ldots, v_n)$, ϕ' denote $\Phi(v'_1, \ldots, v'_n)$ and so on.

The following are properties that aggregation rules may satisfy.[18]

Independence. For all $p \in X$ and all $(v_1, \ldots, v_n), (v'_1, \ldots, v'_n) \in V^n$, $v_i(p) = v'_i(p)$ for all $i \in N$ implies $\phi(p) = \phi'(p)$.

Systematicity. For all $p, q \in X$ and all $(v_1, \ldots, v_n), (v'_1, \ldots, v'_n) \in V^n$, $v_i(p) = v'_i(q)$ for all $i \in N$ implies $\phi(p) = \phi'(q)$.

Unanimity. For all $v \in V$, $\Phi(v, \ldots, v) = v$.

Veto-dictatorial. There exists some $i \in N$ such that for all $p \in X$ and every $(v_1, \ldots, v_n) \in V^n$, $v_i(p) = 0$ implies $\phi(p) = 0$.

An aggregation rule is said to be free from veto-dictatorship if it is not veto-dictatorial.[19]

Our definition of a veto-dictatorial aggregation rule is weaker than the standard definition of a dictatorial rule. The latter says that, at every profile, the collective judgments are identical to one individual’s judgments.

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[18] Mongin [16] is an important discussion of the properties of aggregation rules.

[19] Our veto-dictatorship condition comes from Pauly and van Hees [19].

3 Theorems

3.1 Results

We first give a formal definition of the averaging rule that we discussed earlier. Define a function Γ with domain V^n as follows. Let us denote $\Gamma(v_1, \ldots, v_n)$ by γ. That is, γ is the output of Γ at profile $(v_1, \ldots, v_n)$. For each profile $(v_1, \ldots, v_n) \in V^n$, let γ be a function from X to $[0, 1]$ such that $\gamma(p) = \frac{1}{n} \sum_{i \in N} v_i(p)$ for all $p \in X$. We call Γ the averaging rule. Our first theorem states that if T is the Łukasiewicz t-norm then the averaging rule is an aggregation rule. That is, the range of Γ is a subset of V, the set of all deductively closed judgments.

Theorem 1. *If T is the Łukasiewicz t-norm then Γ maps from V^n to V.*

Proof. Let Γ again denote the averaging rule. Let T_L denote the Łukasiewicz t-norm. Take any $A \in \mathcal{P}(X)$ and $p \in X$ such that $A \vDash p$, and any profile $(v_1, \ldots, v_n) \in V^n$.

Note that for any $(x_1, \ldots, x_k) \in [0, 1]^k$, $T_L(x_1, \ldots, x_k) = \max(0, 1 - k + \sum_{i=1}^{k} x_i)$. So we have $T_L(v_{iA}) = \max(0, 1 - |A| + \sum_{a \in A} v_i(a))$ for all $i \in N$. Given that every individual's judgment is deductively closed, we know that $T(v_{iA}) \leq v_i(p)$ for all $i \in N$. It follows that $1 - |A| + \sum_{a \in A} v_i(a) \leq v_i(p)$ for all $i \in N$. So, summing both sides of this inequality over N, we have $(1 - |A|)n + \sum_{i \in N} \sum_{a \in A} v_i(a) \leq \sum_{i \in N} v_i(p)$. And dividing both sides by n gives us $1 - |A| + \frac{1}{n} \sum_{i \in N} \sum_{a \in A} v_i(a) \leq \frac{1}{n} \sum_{i \in N} v_i(p)$. By definition, $\gamma(p) = \frac{1}{n} \sum_{i=1}^{n} v_i(p)$ and $\gamma(a) = \frac{1}{n} \sum_{i=1}^{n} v_i(a)$ for all $a \in A$, so we can write $1 - |A| + \sum_{a \in A} \gamma(a) \leq \gamma(p)$.

By definition, $T_L(\gamma_A) = \max(0, 1 - |A| + \sum_{a \in A} \gamma(a))$. Given that $\gamma(p)$ cannot be less than zero, and our finding that $1 - |A| + \sum_{a \in A} \gamma(a) \leq \gamma(p)$, we see that $T_L(\gamma_A) \leq \gamma(p)$. $\square$

The first lemma establishes the following fact, which we appeal to in the proof of Lemma 3. Take any value $x \in [0, 1]$ and any subset A of the agenda. Suppose that an individual assigns the value 1 to every proposition that is entailed by A and assigns x to every other proposition. Then this individual's judgment is deductively closed. We denote such a judgment by $f^{(A)}$.

Lemma 2. *For all $A \in \mathcal{P}(X)$ and all $x \in [0, 1]$, the judgment $f^{(A)}$ defined as follows is deductively closed. For all $p \in X$, if $A \vDash p$ then $f^{(A)}(p) = 1$, otherwise $f^{(A)}(p) = x$.*

Proof. Take any $A \in \mathcal{P}(X)$. The judgment $f^{(A)}$ is deductively closed *unless* there is $p \in X$ and $B \in \mathcal{P}(X)$ such that $B \vDash p$ and $T(f_B^{(A)}) > f^{(A)}(p)$. Let us assume that such a p and B do exist. Since the range of $f^{(A)}$ is just $\{x, 1\}$, it follows that $f^{(A)}(p) = x$. Recalling the definition of $f^{(A)}$, we have then $A \nvDash p$.

Since T is a t-norm we know that for any k-tuple $(x_1, \dots, x_k) \in [0,1]^k$, $T(x_1, \dots, x_k)$ is no greater than the least component of $(x_1, \dots, x_k)$. To see that this is the case, note that for all $y, z \in X$, $y \leq z$, properties (iii) and (iv) of T imply $T(y, z) \leq T(y, 1) = y$.

So $T(f_B) > x$ implies that $f^{(A)}(b) > x$ for all $b \in B$. Since the range of $f^{(A)}$ is just $\{x, 1\}$, it follows that $f^{(A)}(b) = 1$ for all $b \in B$. Therefore, by our definition of $f^{(A)}$, it must be that $A \vDash b$ for all $b \in B$. However, by condition (2*) on the entailment relation, if $A \vDash b$ for all $b \in B$, and $B \vDash p$, then $A \vDash p$. This contradicts $A \nvDash p$. $\square$

The following lemma derives systematicity from independence and unanimity, as is common in the literature. This derivation requires that X is totally blocked.[20]

Lemma 3. *Assume that X is totally blocked and that $\Phi : V^n \to V$ is independent and unanimous. Then Φ is systematic.*

Proof. Let Φ be an aggregation rule that is independent and unanimous.

This proof is in two parts. In the first part we establish that, for all profiles $(v_1, \dots, v_n) \in V^n$ and all $p, q \in X$ such that $p \rhd q$, if $v_i(p) = v_i(q)$ for all $i \in N$ then $\phi(p) \leq \phi(q)$. This is true whether X is totally blocked or not. In the second part we use this fact to demonstrate that if X is totally blocked then Φ must be systematic.

Take a pair of propositions $p, q \in X$ such that $p \rhd q$.

Case 1. Suppose that $\{p\} \vDash q$. It follows that for any $(v_1, \dots, v_n) \in V^n$, $\phi(p) \leq \phi(q)$ since ϕ is in V.

Case 2. Suppose that there exists $A \in \mathcal{P}(X)$ such that $(A \cup \{p\}) \vDash q$ while $A \nvDash q$. Take any profile $(v_1, \dots, v_n) \in V^n$ such that $v_i(p) = v_i(q)$ for all $i \in N$.

Let $(v'_1, \dots, v'_n) \in V^n$ be the profile where, for all $i \in N$ and all $r \in X$, if $A \vDash r$ then $v'_i(r) = 1$, and $v'_i(r) = v_i(p)$ otherwise. Lemma 2 shows that this profile exists.

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[20] Lemma 3 is similar to Theorem 2 in Dietrich and List [7]. However, since our models differ it is important to provide an independent statement and proof of this result.

Condition (1*) of the entailment relation requires that $A \vDash a$ for all $a \in A$. Therefore, from what we know about $(v'_1, \ldots, v'_n)$, we have $v'_i(a) = 1$ for all $a \in A$ and all $i \in N$. If it were also true that $A \vDash p$ then that would mean that $A \vDash s$ for all $s \in A \cup \{p\}$. Condition (2*) would then imply that $A \vDash q$. Since this would be a contradiction it must be that $A \nvDash p$. So, given that $A \nvDash p$ and $A \nvDash q$, we have $v'_i(p) = v'_i(q) = v_i(p)$ for all $i \in N$.

Unanimity and independence imply that $\phi'(a) = 1$ for all $a \in A$. Since ϕ'_A is an $|A|$-tuple of ones and t-norms have the property that 1 is a neutral element, we have $T(\phi'_A, \phi'(p)) = \phi'(p)$. Since $(A \cup \{p\}) \vDash q$, deductive closure requires that $T(\phi'_A, \phi'(p)) \leq \phi'(q)$ and so $\phi'(p) \leq \phi'(q)$. It must also be true, since Φ is independent, that $\phi(p) \leq \phi(q)$.

We can use this argument to show that if X is totally blocked then Φ is systematic.

Assume that X is totally blocked. We know then that for all $r, s \in X$ either $r \vartriangleright s$ or there are $p_1, \ldots, p_k \in X$ such that $r \vartriangleright p_1 \vartriangleright \ldots \vartriangleright p_k \vartriangleright s$. By applying the above argument repeatedly we can see that for any profile $(v''_1, \ldots, v''_n) \in V^n$ such that $v''_i(r) = v''_i(s)$ for all $i \in N$, it must be the case that $\phi''(r) \leq \phi''(s)$. Similarly, either $s \vartriangleright r$ or the agenda must contain $q_1, \ldots, q_k$ such that $s \vartriangleright q_1 \vartriangleright \ldots \vartriangleright q_k \vartriangleright r$ and so $\phi''(s) \leq \phi''(r)$. Hence $\phi''(r) = \phi''(s)$. □

We are now in a position to state our central theorem.

Theorem 4. *If T has a zero divisor then there exists an aggregation rule $\Phi : V^n \to V$ that is independent, unanimous and free from veto-dictatorship. If T has no zero divisor and X is non-trivially, totally blocked then no such aggregation rule exists.*

Proof. We first prove that if T has no zero divisor and X is non-trivially, totally blocked then every aggregation rule that is independent and unanimous is veto-dictatorial.

Assume that X is non-trivially, totally blocked. Take any $p, q \in X$ such that $p \vartriangleright q$ and $\{p\} \nvDash q$. There must exist $A \in \mathcal{P}(X)$ such that $(A \cup \{p\}) \vDash q$ while $B \nvDash q$ for all non-empty $B \subsetneq A \cup \{p\}$. Let r denote an element of A such that $\{p\} \nvDash r$ and let Z denote $A - \{r\}$. If Z is empty then ignore the references to it in the tables below.

Take a profile $(v_1, \ldots, v_n) \in V^n$ such that $v_i(p) = 0$ for all $i \in N$. Unanimity and independence imply that $v(p) = 0$. Take some other profile

$(\hat{v}_1, \ldots, \hat{v}_n) \in V^n$ such that $\hat{v}_i(p) = 1$ for all $i \in N$. By an identical argument, $\hat{v}(p) = 1$. Consider the following sequence of profiles.

$$\begin{aligned}
\mathbf{W}^{(0)} &= (v_1, ..., v_n),\\
\mathbf{W}^{(1)} &= (\hat{v}_1, v_2, .., v_n),\\
\mathbf{W}^{(2)} &= (\hat{v}_1, \hat{v}_2, v_3, .., v_n),\\
&...\\
\mathbf{W}^{(n)} &= (\hat{v}_1, ..., \hat{v}_n).
\end{aligned}$$

Let $\phi^{(0)}$ denote $\Phi(\mathbf{W}^{(0)})$, $\phi^{(1)}$ denote $\Phi(\mathbf{W}^{(1)})$ and so on. At some profile in this sequence the collective judgment on p must rise from zero to a value strictly greater than zero. Assume, without loss of generality, that this happens at $\mathbf{W}^{(2)}$.

We can construct a profile $(v'_1, \ldots, v'_n) \in V^n$ where individuals hold the following judgments.

	p	q	r	Z
Individual 1	1	1	1	1
Individual 2	0	0	1	1
Everyone else	1	0	0	1

Since X is totally blocked, we know from Lemma 3 that Φ is systematic. Systematicity implies that $\phi'(r) = \phi^{(2)}(p)$, and so we have $\phi'(r) > 0$. Systematicity also implies that $\phi'(q) = \phi^{(1)}(p)$, and so $\phi'(q) = 0$.

We can write $T(\phi'_A)$ as $T(\phi'_Z, \phi'(r))$. Since ϕ'_Z is just a $|Z|$-tuple of ones, and 1 is a neutral element of a t-norm, it must be the case that $T(\phi'_Z, \phi'(r)) = \phi'(r)$. So we have $T(\phi'_A) = \phi'(r) > 0$.

Recall that $(A \cup \{p\}) \vDash q$ and so $T(\phi'_A, \phi'(p)) \leq \phi'(q)$. Since $\phi'(q) = 0$ it must be the case that $T(\phi'_A, \phi'(p)) = 0$. Given that $T(\phi'_A) > 0$ and T has no zero divisor, it must be that $\phi'(p) = 0$.

We have seen that $\phi'(p) = 0$ despite the fact that $v'_i(p) = 1$ for all $i \in N - \{2\}$. The proof of veto-dictatorship can be completed as follows.

We have assumed from the beginning that $p \rhd q$. It is either true or false that $q \rhd p$. We examine these two cases. Let $(t_1, \ldots, t_n)$ be any element of $[0, 1]^n$.

Case 1. Assume it is false that $q \rhd p$. Consider a profile $(v''_1, \dots, v''_n) \in V^n$ where individuals hold the following judgments.

	p	q	r	Z
Individual 1	t_1	1	1	1
Individual 2	0	0	1	1
Individual 3	t_3	0	0	1
...				
Individual n	t_n	0	0	1

Systematicity implies that $T(\phi''_A) = T(\phi'_A)$, so we have $T(\phi''_A) > 0$. Systematicity also implies that $\phi''(q) = \phi'(q)$, so $\phi''(q) = 0$. Since $(A \cup \{p\}) \vDash \{q\}$ we have $T(\phi''_A, \phi''(p)) \leq \phi''(q)$. As T has no zero divisor, it must be the case that $\phi''(p) = 0$.

Case 2. Assume it is true that $q \rhd p$. There then exists a subset $C \subseteq X$ such that $C \cup \{q\} \vDash p$ with $C \nvDash p$ or $C = \emptyset$. If C is empty then ignore the references to it in the table below. Consider a profile $(v^*_1, \dots, v^*_n) \in V^n$ where individuals hold the following judgments.

	p	q	C
Individual 1	1	t_1	1
Individual 2	0	0	1
Individual 3	1	t_3	1
...			
Individual n	1	t_n	1

Independence implies that $\phi^*(p) = \phi'(p) = 0$. Additionally, due to the fact that 1 is a neutral element of a t-norm or the fact that C is empty, we know that $T(\phi^*_C, \phi^*(q)) = \phi^*(q)$. Since $C \cup \{q\} \vDash p$, it must be true that $T(\phi^*_C, \phi^*(q)) \leq \phi^*(p)$ and so $\phi^*(q) \leq \phi^*(p)$. In other words, $\phi^*(q) = 0$.

The desired result now follows from systematicity.

We now prove that there exists an aggregation rule that is independent, unanimous and not veto-dictatorial if T has a zero divisor. Here we drop the assumption that X is non-trivially, totally blocked.

First, note that if T has a zero divisor then, since T is monotonic and commutative, there must exist $z \in (0,1)$ such that $T(z,z) = 0$. Let us fix such a number z.

For brevity we write $\max(p)$ and $\min(p)$ as shorthand for the greatest and least components respectively of $(v_1(p), \ldots, v_n(p))$.

Define an aggregation rule Ψ as follows. For all $(v_1, \ldots, v_n) \in V^n$ and all $p \in X$, let $\psi(p)$ be equal to the median of the three numbers z, $\max(p)$ and $\min(p)$. It is clear that Ψ is independent and unanimous. We need to prove that for all $(v_1, \ldots, v_n) \in V^n$, ψ is deductively closed.

Assume, by way of contradiction, that ψ is not deductively closed. Then there exists $p \in X$, $A \in \mathcal{P}(X)$ and $(v_1, \ldots, v_n) \in V^n$ such that $A \vDash p$ and $T(\psi_A) > \psi(p)$.

From the definition of Ψ we can see that for all propositions $q \in X$, $\psi(q) \geq \min(q)$. Hence, one of the following three statements must be true.

1. For all $a \in A$, $\psi(a) = \min(a)$.
2. There are at least two distinct propositions $a_1, a_2 \in A$ such that $\psi(a_1) > \min(a_1)$ and $\psi(a_2) > \min(a_2)$.
3. There is one, and only one, proposition $a_1 \in A$ such that $\psi(a_1) > \min(a_1)$.

We consider each of the above three cases in turn.

Case 1. Since $\psi(a) \leq v_i(a)$ for all $a \in A$, it follows from the monotonicity of T that $T(\psi_A) \leq T(v_{iA})$ for all $i \in N$. Given our assumption that $T(\psi_A) > \psi(p)$ and the fact that individual judgments are deductively closed, it must be the case then that $\psi(p) < T(\psi_A) \leq T(v_{iA}) \leq v_i(p)$ for all $i \in N$. So $\psi(p) < v_i(p)$ for all $i \in N$. However, from the definition of Ψ we can see that it is not possible for $\psi(p)$ to be strictly less that $\min(p)$. So we have a contradiction.

Case 2. We have $\psi(a_1) > \min(a_1)$ and $\psi(a_2) > \min(a_2)$. From the definition of Ψ we see that this implies $\psi(a_1) \leq z$ and $\psi(a_2) \leq z$. Given that $T(z,z) = 0$ and that T is monotone, we have $T(\psi(a_1), \psi(a_2)) = 0$. If $A = \{a_1, a_2\}$ then we can write $T(\psi_A) = 0$. Suppose $A \neq \{a_1, a_2\}$. Let B denote the set

$A - \{a_1, a_2\}$. We have $T(\psi_A) = T(\psi_B, \psi(a_1), \psi(a_2)) = T(\psi_B, 0)$. Since T is a t-norm we know that $T(1, \ldots, 1, 0) = 0$ and so, by monotonicity, $T(\psi_B, 0) = 0$. Hence, $T(\psi_A) = 0$. This contradicts our assumption that $T(\psi_A) > \psi(p)$.

Case 3. We have $\psi(a_1) > \min(a_1)$. From the definition of Ψ we see that this implies $\psi(a_1) \leq z$. As T is a t-norm we know that $T(\psi_A)$ cannot be greater than any of the components of the $|A|$-tuple ψ_A. This property of t-norms was identified in the proof of Lemma 2. So $T(\psi_A) \leq z$. By assumption, $T(\psi_A) > \psi(p)$ and so $z > \psi(p)$. Recalling again the definition of Ψ, $z > \psi(p)$ implies $\psi(p) = \max(p)$. Let $i \in N$ be an individual such that $\psi(a_1) \leq v_i(a_1)$. Since $\psi(a) \leq v_i(a)$ for all $a \in A$,[21] it follows by monotonicity that $T(\psi_A) \leq T(v_{iA})$. The valuation v_i is deductively closed so $T(v_{iA}) \leq v_i(p)$. Therefore, $T(\psi_A) \leq v_i(p)$. However, we have seen that $\psi(p) = \max(p)$ and so we have $T(\psi_A) \leq \psi(p)$. This contradicts $T(\psi_A) > \psi(p)$.

In all three cases the assumption that ψ is not deductively closed leads to a contradiction. □

3.2 Interpretation

It is worth explaining the nature of our existence result (the first part of Theorem 4). When T has a zero divisor, there exists an aggregation rule that is independent, unanimous and not veto-dictatorial. Importantly, the agenda is irrelevant here. As we have seen (this is the second part of Theorem 4), if T has no zero divisor then no aggregation rule exists that satisfies our normative requirements when X is non-trivially, totally blocked. These agendas are important (as we have explained, the preference agenda is one such agenda).

It is also important to describe what happens in the model when the set of degrees is required to be $\{0, 1\}$. This is the binary case. In this case, all t-norms are "equivalent" in the sense that, for every tuple of $\{0, 1\}$ values in their domains, the norms generate the same $\{0, 1\}$ value in their ranges.

It is easy to see from the proof of our theorem that when judgment sets are binary, impossibility follows. To state this more precisely, in this case, when X is non-trivially, totally blocked, no aggregation rule exists that is

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[21] This follows from the fact that $\psi(a_1) \leq v_i(a_1)$ and our original assumption about Case 3.

independent, unanimous and free from veto-dictatorship. Not surprisingly, this is in keeping with the (binary) impossibility theorems that have appeared in the literature.[22] The contribution of this short paper is to point out that possibility can arise in a suitably formulated many-valued model (but not all of the time).

To conclude, we should explain how our theorem applies to the classic problem of preference aggregation. For the time being, assume that we are operating in a binary world (this is the dominant approach in the literature). We demonstrated in Section 2.1 that a preference agenda is a non-trivially, totally blocked agenda. Our model requires that individual and collective judgments are deductively closed. That assumption corresponds, in the theory of preference aggregation, to the requirement that individual and collective preferences are transitive, but not necessarily complete.

To illustrate, consider a judgment where $f(aRb) = 1$ and $f(p) = 0$ for all $p \in \{aRb, aRc, bRc, bRa, cRa, cRb\} - \{aRb\}$. This judgment is deductively closed since aRb does not entail any of the other propositions. However, f does not correspond to any complete ordering of the three alternatives. As described above for the general binary case, no aggregation rule exists that is unanimous, independent and free from veto-dictatorship.[23] However, when judgments on preferences are many-valued, Theorem 4 says that impossibility does not necessarily follow.

In this setting, our requirement of deductive closure (condition (1)) works like a transitivity condition for many-valued preferences, with

$$T(f(aRb), f(bRc)) \leq f(aRc) \text{ given that } \{aRb, bRc\} \vDash aRc.^{24}$$

If T has a zero divisor, then despite the fact that the agenda is non-trivially, totally blocked, preference aggregation is possible under our normative requirements. However, as we have seen, this is impossible when T has no zero divisor.

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[22] See Dietrich and List [6] and Dokow and Holzman [9].

[23] This is in keeping with the results of Weymark [23] and Dokow and Holzman [9] for the binary case. They express their impossibility theorems in terms of the existence of an "oligarchy".

[24] This condition is called "max-star" transitivity in the literature on social choice with many-valued preferences. See Duddy, Perote-Peña and Piggins [11].

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