Contracts and Coalition Formation based on Individual Deviations*

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Abstract

This paper revisits the work of Dreze and Greenberg (1980) in which endogenous coalition formation problems are studied based on three stability notions that reflect three different contractual arrangements with respect to one-person deviations. Analyzing coalition formation within the framework of transferable utility games, we propose to modify the original definitions of these stability notions due to some, in our opinion, unwanted characteristics. To emphasize this difference we re-baptize the stability notions into individual stability, contractual stability, and compensation stability. In particular, with the new definition individual rationality is implied by individual stability. We, furthermore, show that any coalition structure of maximum social worth is both contractually and compensation stable. In addition, we analyze existence and study the relations between the stability notions. We show that an individually stable outcome is also contractually stable and that an individually rational and compensation stable outcome is also contractually stable.

Applying the general framework to an example of mutual insurance in production, we find that in each type of contractual setting there are stable individually rational pooling outcomes while, on the contrary, individually rational separating outcomes are not stable in general.

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1 Introduction

Dreze and Greenberg (1980) introduced the notions of contractual arrangements in the analysis of endogenous coalition formation problems. Their framework is thus well suited to answer the simultaneous question posed by Aumann and Dreze (1974): in a setting in which players face a cooperative game, which coalitions will form, and how will their members split the proceeds? Dreze and Greenberg (1980) were also the first to define the notions of individual and contractual stability in a setting in which transfers between groups in a coalition structure are allowed. An individual (a coalition) may compensate her former coalition members (the individual who leaves the coalition) in case that the change makes them (her) worse-off.

However, to the best of our knowledge the literature spawned by their seminal paper, *e.g.*, Banerjee, Konishi and Sonmez (2001), Bogomolnaia and Jackson (2002), and Pápai (2004), abstracted from the problem of value allocation. In these works, it is assumed that each player's preferences over membership in all the coalitions of which she can be a member are exogenously determined .

In this paper, we offer formulations of the notions of individual, contractual, and compensation stability that are modifications of the ones introduced by Dreze and Greenberg (1980). We should point out that there are some notable differences between the setting analyzed by Dreze and Greenberg (1980) and our setting. In particular, the characteristic functions analyzed in Dreze and Greenberg (1980) are derived from a production setting with both private and public goos together with a hedonic aspect in relationships between players. In our work, instead, we consider exogenously given unrestricted characteristic value functions that may also assume negative values. The main problem with the original notions of stability in our setting is that only deviations are considered that lead to the *same number* of coalition structure elements. This type of restriction implies, for example, that in a coalition formation problem, the grand coalition, and *any* feasible payoff allocation will be individually stable since a deviation of a single player to the empty set is not allowed. Our alternative definition of individual stability makes it explicit that players might consider to join another coalition structure element, or the empty set, based on their marginal contribution. Thus in our setting, a player may only join another coalition, if all the members of the new coalition receive a strictly higher payoff than before irrespective of the size of the new coalition relative to the old one. Furthermore, by allowing players to deviate by 'joining the empty set', the definition of individual stability implies individual rationality. Individual rationality, we think, is a natural requirement in an equilibrium concept relevant for a setting in which players are not restricted in their coalitional membership by binding contracts, as players should have the ability to opt out of a coalition and be alone if by doing so they obtain higher payoff.

The aim of our paper is to revive the interest in studying endogenous coalition formation problems under various contractual arrangements. This approach is applicable to environments in which contracts are enforceable. We regard the coalition[3] and an individual member as two sides in a contract. The arrangement gives particular rights, such as the right to end the contract and the right to be compensated to one or both of the parties. Depending on the allocation of rights, we distinguish between three types of contractual arrangements leading to the notions of *individual, contractual,* and *compensation* stability. Individual and contractual stability give the individual member the right to end the contract but do not allow for compensation rights. Contractual stability setting in addition gives a veto right to the coalition. To end the contract a member needs the agreement of the rest of the coalition members. If they are worse-off after the change, they disagree. The third concept, compensation stability, gives both parties the right to end the contract and the right to be compensated in case they are worse-off after the change. Compensation stability has two complementary sides to it: pull-in and push-out stability. Pull-in stability reflects external stability by allowing a coalition to attract new members. On the other hand, push-out stability reflects internal stability by allowing coalitions to push out one of their members.

The focus on contractual arrangements is justified, on the one hand, by the fact that contracts are common arrangements in economic organization in which economic value is generated by the cooperation of group members, *e.g.*, business alliances and consumer clubs. Moreover, it is motivated by the observation that agents, being part of large groups, are anonymous to each other. Thus, no group deviations are allowed in our setting. This makes other well established equilibrium notions, such as the core, inadequate for our stability analysis.

To make this point more clear, we elaborate on the behavioral assumptions we make. We assume that each player makes a decision to join/leave a coalition based on her own perceived payoff, without taking into account the effect of her actions on other players' payoffs. A

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[3] Here by coalition we mean the rest of the coalition members.

player's payoff from a coalition membership is *endogenously* determined. It depends on her "power" to obtain a share of the coalition value. This power-based measure, though related, is not entirely determined by the player's marginal contribution to the coalition value as it also depends on her outside options, *i.e.*, what other coalitions are being formed and how much she can possibly get as a share of their payoff, *i.e.*, the *endogenous coalition structure*. In this way a member of a coalition with high value might prefer to switch to a coalition with a smaller value because her payoff in the latter is higher than the share she gets from the former. By switching coalitions, though, a player changes the outside options for the rest of the players, which she does not take into account at the time of the move. Accordingly, by switching coalitions, the player also affects another player's relative power in bargaining for a share from the coalition value, not only in the two coalitions in which the membership has changed, but also in the rest of the coalitions in the coalition structure. Not taking into accounts the market-wide effects of her actions makes each player myopic.

As an application of our theoretical framework we consider group cooperation driven by mutual insurance. The value of a cooperating group is generated by the ability of group members to smoothen production by pooling the risk of an idiosyncratic shock. The objective of a mutual insurance group is to maximize the total output of the group members. In this setting the members of an insurance group are stakeholders. As such they divide the value of the group generated by their cooperation amongst themselves.[4] Credit union and building societies are such types of insurance groups in which the members are stakeholders. Some sustainable development programs encourage agricultural producers to form mutual insurance groups to insure against agricultural yield damage.[5] To illustrate the risk of a negative shock, we adopt the model studied by Rothschild and Stiglitz (1976). We refer to our setting, however, as "*mutual* insurance" to emphasize the cooperative nature of the problem and distinguish it from the third-party market insurance setting studied by Rothschild and Stiglitz (1976). We focus on the stability of two types of outcomes widely studied in the literature: pooling and separating outcomes. We find that given the assumption of risk-averse players, all pooling outcomes are contractually and compensation stable, while no separating outcomes are individually or compensation stable. The individually rational pooling outcomes

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[4] The advantages of employing a cooperative approach to studying insurance organizations has been discussed by Boyd, Prescott and Smith (1988).

[5] See Sperling, Osborn and Cooper (2004) for a discussion on seed security and policies concerning local communities.

are individually stable and, moreover, this type of outcomes exists in every mutual insurance formation problem. Finally, no individually rational separating outcome is contractually stable either. What drives these results is the possibility of side-payments within groups and, in the case of the compensation setting, between groups.

The remainder of the paper is structured in the following way. In Section 2 we define the coalition formation problem and present the three stability concepts. We present existence results as well as a technical discussion on the relation between the concepts in Section 3. In Section 4 we apply the problem to mutual insurance groups formation and discuss the stable outcomes in terms of the risk composition of the groups.

2 Contractual Settings

There is a finite set of players $N = \{1, 2, \dots, n\}$. Players in the set N form coalitions. The collection of subsets of N is denoted by 2^N. A player cannot be a member of more than one coalition. A partition of N into non-empty coalitions forms a *coalition structure* and it is denoted by P. The set of all possible coalition structures is denoted by $\mathcal{P}$. Each coalition generates value by the cooperation of its members. This value differs depending on the identity of the group members. For the purposes of the general analysis, it suffices to regard this relation as given by a value function $v : 2^N \to \mathbb{R}$. The pair (N, v) such that $v(\emptyset) = 0$ defines a coalitional game. Without loss of generality, we study zero-normalized problems, *i.e.*, $v(\{i\}) = 0$ for all $i \in N$.

An outcome of a coalitional game is represented by a *payoff configuration*. A payoff configuration is a pair (P, x) where $P \in \mathcal{P}$ is a coalition structure of N and $x \in \mathbb{R}^N$ is an efficient payoff vector for P, *i.e.*, $x(S) = v(S)$ for all $S \in P$, where $x(S) := \sum_{i \in S} x_i$.

We refer to a coalition structure P which has a maximum total coalition structure value as a coalition structure of *maximum social worth*, *i.e.*, $\sum_{S \in P} v(S) \geq \sum_{S \in P'} v(S)$ for all coalition structures $P' \in \mathcal{P}$. Similarly the payoff configuration (P, x) such that P is coalition structure of maximum social worth and x is efficient is called a payoff configuration *maximum social worth*.

Given the coalition value, a player's payoff depends on an endogenous allocation of the group value. A player $i \in N$ prefers to be a member of a coalition which yield a higher payoff to her. A fair player's payoff in a coalition will also depend on the exact coalition structure of the set of players, since the composition of the coalition structure defines the outside options. This point will become more clear with the discussion of the stability concepts below.

2.1 Individual Stability

In the contractual arrangement of individual stability the right to end the contract is given only to the individual players and no compensatory obligations are imposed. Individual stability thus entails that a player cannot obtain a higher payoff by joining another coalition structure element, or by forming a singleton coalition.[6]

Definition 2.1 *Let* (N, v) *be a coalitional game. A payoff configuration* (P, x) *is* ***individually stable*** *if there are no* $i \in N$ *and* $S \in P \cup \{\emptyset\}$ *with* $i \notin S$ *such that*

$$x_i < v(S \cup \{i\}) - v(S).$$

The following example is used to illustrate the concept of individual stability.

Example 2.2 Let $N = \{1, 2, 3\}$, $v(\{1, 2\}) = 2$, $v(\{1, 3\}) = 3$, $v(\{2, 3\}) = 4$, and $v(S) = 0$, otherwise.

In the given coalitional game there is only one individually stable payoff configuration, namely, $(\{N\}, 0)$. Clearly, this is an individually stable outcome: the players can deviate only by joining the empty set and obtain a payoff of zero.

No other payoff configuration is individually stable because there is always at least one player who wants to deviate. As an example consider the payoff configurations of maximum social worth $\big(\{\{1\}, \{2, 3\}\}, (0, \alpha, 4 - \alpha)\big)$ with $\alpha \in \mathbb{R}$. The best outside option for player 2 is to join 1 in the coalition $\{1, 2\}$ where player 2's marginal contribution is two. Thus for player 2 not to have incentives to deviate it must be that $x_2 = \alpha \geq 2$. Similarly, for player 3 not to deviate, it must hold that $x_3 = 4 - \alpha \geq 3$. The two conditions cannot hold simultaneously in any efficient payoff vector and hence this type of payoff configurations cannot be individually stable. ♦

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[6] Implicit in the definition of individual stability is that a player can join a coalition only if her membership is unanimously approved by the current coalition members. This is to say, a player can join a coalition if the current members have at least as high payoff after she joins as they had before. This is why when a player decides on joining a coalition she bases her decision on her marginal contribution to the coalition value.

As seen above, there are coalitional games in which no payoff configuration of maximum social worth is individually stable. The next example shows that there are coalitional games in which there are no individually stable payoff configurations.

Example 2.3 Let $N = \{1, 2, 3\}$, $v(\{1, 2\}) = 2$, $v(\{1, 3\}) = 3$, $v(\{2, 3\}) = 4$, $v(N) = -1$, and $v(S) = 0$, otherwise.

Note that the only difference with Example 2.2 is that here the grand coalition has a negative value. The payoff configuration that consists of the grand coalition and an efficient payoff vector cannot be individually stable: by the efficiency of the payoff vector follows that there is at least one player who has a negative payoff, thus, such player will deviate by forming a singleton coalition. ♦

2.2 Contractual Stability

Contractual stability is based on individual incentives under the additional condition that a deviating player needs to acquire permission from the coalition, whose member she is, in case she wants to end the contract. The coalition grants the permission only if the rest of the coalition members are as well-off without that particular player as when she is part of the coalition. In a contractually stable payoff configuration no player can obtain a higher payoff by joining another coalition structure element without making the members of her current coalition worse-off.

Definition 2.4 *Let* (N, v) *be a coalitional game. A payoff configuration* (P, x) *is* ***contractually stable*** *if there are no* $i \in N$ *and* $S,T \in P \cup \{\emptyset\}$ *with* $i \in T$ *and* $S \neq T$ *such that*

$$x_i < v(S \cup \{i\}) - v(S) \quad \textit{and} \quad x(T \setminus \{i\}) \leq v(T \setminus \{i\}).$$

The next example illustrates how contractual stability limits the deviating possibilities of a player in contrast to individual stability.

Example 2.5 Let (N, v) be the coalitional game of Example 2.2.

There are infinitely many contractually stable payoff configurations in this coalitional game. Any coalition structure with the exclusion of the coalition structure that consists of all singletons can be part of a contractually stable payoff configuration. Consider the following type of coalition structures: $\big(\{\{1,3\},\{2\}\},(\alpha,0,3-\alpha)\big)$ with $\alpha \in (0,3]$. These are contractually stable payoff configurations: player 2's outside option is to join $\{1,3\}$ in the grand coalition but there her marginal contribution is negative. Neither player 1 nor 3 will give a permission to the other who has incentives to join player 2 since by being alone that player will have a strictly lower payoff.

A payoff configuration that is not contractually stable is $\big(\{\{1,3\},\{2\}\},(0,0,3)\big)$. Player 3's outside option to join player 2 in coalition $\{2,3\}$ gives her a higher payoff than what she has, *i.e.* $v(\{2,3\}) - v(\{2\}) = 4 > 3 = x_3$. Player 1 grants permission to player 3 to leave as she is indifferent between having a payoff of zero in the coalition $\{1,3\}$ or as a member of the singleton coalition $\{1\}$. ♦

The above example shows that in the contractual stability setting, there are situations in which a one-person deviation can lead to an increase in the total value of the coalition structure, yet, it is not performed because one of the parties of the contract is strictly worse-off. To overcome this restriction on profitable deviations, in the next contractual specification we allow for side payments between players after the contract between them has ended .

2.3 Compensation Stability

First, the two complementary sides of compensation stability, pull-in stability and push-out stability, are introduced.

In the contractual setting of pull-in stability, the individual player is the only party who can end the contract. In case the remaining group members are worse-off, the new coalition of the deviating player is obliged to compensate them for this loss. Thus, in a pull-in stable outcome there is no coalition structure element that by attracting a new member may increase its value enough to give higher payoffs to its members after possible compensation of the incoming player's previous coalition.

Definition 2.6 *Let (N,v) be a coalitional game. A payoff configuration (P,x) is **pull-in stable** if there are no $i \in N$ and $S,T \in P$ with $i \in T$ and $S \neq T$ such that*[7]

$$v(S) \quad < \quad v(S \cup \{i\}) - x_i - \max\Big\{0, x(T \setminus \{i\}) - v(T \setminus \{i\})\Big\}.$$

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[7] Since the empty set is not regarded as a coalition structure element, it is not included in the possible set of coalitions that can pull a player in.

In push-out stability the right to end the contract is given only to the coalition. A coalition wants to end the contract with one of its members if by doing so, it can increase the payoffs of the remaining members. In the push-out setting a compensation is required in case the member whose contract has been ended has a lower best outside option than her current payoff as a coalition member.

Definition 2.7 *Let* (N, v) *be a coalitional game. A payoff configuration* (P, x) *is* ***push-out stable*** *if there are no* $i \in N$ *and* $S, T \in P \cup \{\emptyset\}$ *with* $i \in T$ *and* $S \neq T$ *such that*

$$x(T \setminus \{i\}) \quad < \quad v(T \setminus \{i\}) - \max\Big\{0, x_i - \big(v(S \cup \{i\} - v(S))\big)\Big\}.$$

Combining pull-in and push-out stability, we have compensation stability.

Definition 2.8 *An outcome of a coalitional game is* ***compensation stable*** *if it is pull-in and push-out stable.*

To illustrate that compensation setting may overcome the restriction on profitable deviations of the contractual stability setting, we consider the following example.

Example 2.9 Let (N, v) be as in Example 2.2.

There is one type of compensation stable payoff configurations, *i.e.*, the payoff configurations of maximum social worth $\big(\{\{1\}, \{2, 3\}\}, (0, \alpha, 4 - \alpha)\big)$ with $\alpha \in \mathbb{R}$.

These payoff configurations are pull-in stable. Coalition $\{2, 3\}$ does not want to attract player 1 as a member since the grand coalition has lower value than their current value. Player 1 can increase the value of its coalition by attracting either player 2 or 3. Yet, the increase is not enough to give her a higher payoff after she compensates the remaining member of coalition $\{2, 3\}$ for the change. These payoff configurations are also push-out stable. Neither player 2 nor 3 can increase her payoff by pushing the other player out to join player 1 in a coalition and compensate her for the change, if needed. Player 1 cannot be pushed out of the singleton coalition either. ♦

3 Existence and Relations

The discussion of existence and relations between the stability contracts is focused on the notions of compensation and contractual stability. Example 2.3 shows that there are coalitional games with no individually stable payoff configurations.

From the definitions of the stability concepts the following results can be obtained in a straightforward fashion.

Proposition 3.1 *Let* (N, v) *be a coalitional game. Then the following results hold:*

(i) Any individually stable payoff configuration is contractually stable;

(ii) Any individually stable payoff configuration is pull-in stable;

(iii) Any payoff configuration $(\{N\}, x)$ *is pull-in stable;*

(iv) Any payoff configuration $(\{i\}_{i \in N}, 0)$ *is push-out stable.*

We establish positive existence results with respect to compensation and contractual stability. In particular, all payoff configurations of maximum social worth are compensation and contractually stable.

Theorem 3.2 *Any payoff configuration of maximum social worth is both compensation and contractually stable.*

Proof. Let (N, v) be a coalitional game. Let $(P^\star, x)$ be a payoff configuration of maximum social worth of (N, v).

Compensation stability: Suppose $(P^\star, x)$ is not a compensation stable payoff configuration. Then either $(P^\star, x)$ is not pull-in stable or $(P^\star, x)$ is not push-out stable.

First, suppose $(P^\star, x)$ is not a pull-in stable payoff configuration. Then there are $i \in N$ and $S, T \in P^\star$ with $i \in T$ and $S \neq T$ such that

$$v(S) < v(S \cup \{i\}) - x_i - \max\Big\{0, x(T \setminus \{i\}) - v(T \setminus \{i\})\Big\}.$$

Using the efficiency of the payoff vector, the above inequality implies

$$v(S) + v(T) < v(S \cup \{i\}) + v(T \setminus \{i\}).$$

So the coalition structure $P = \Big[P^\star \setminus \{S, T\}\Big] \cup \{S \cup \{i\}, T \setminus \{i\}\}$ has a higher total value contradicting that $P^\star$ is a coalition structure of maximum social worth.

Now suppose $(P^\star, x)$ is not a push-out stable outcome. Then there are $i \in N$ and $S, T \in P^\star \cup \{\emptyset\}$ with $i \in T$ and $S \neq T$ such that

$$x(T \setminus \{i\}) < v(T \setminus \{i\}) - \max\Big\{0, x_i + \big(v(S \cup \{i\}) - v(S)\big)\Big\}.$$

Using the efficiency of the payoff vector, the above inequality implies

$$v(T) + v(S) < v(T \setminus \{i\}) + v(S \cup \{i\}),$$

establishing a contradiction.

Contractual stability: Suppose $(P^\star, x)$ is not a contractually stable outcome. Then there are $i \in N$ and $S, T \in P^\star \cup \{\emptyset\}$ with $i \in T$ and $S \neq T$ such that

$$\begin{array}{rcl} x_i & < & v(S \cup \{i\}) - v(S) \\ x(T \setminus \{i\}) & \leq & v(T \setminus \{i\}). \end{array}$$

Adding up the two inequalities and using the efficiency of the payoff vector, we find

$$v(T) + v(S) < v(T \setminus \{i\}) + v(S \cup \{i\}),$$

establishing a contradiction. ■

For establishing a relation between compensation stability and contractual stability we need to introduce one additional property. A payoff configuration (P, x) is *individually rational* if $x_i \geq v(\{i\})$ for all $i \in N$. For zero-normalized problems the individual rationality property is given by $x_i \geq 0$ for all $i \in N$

Proposition 3.3 *Any compensation stable payoff configuration which is individually rational is also contractually stable.*

Proof. Let (N, v) be a coalitional game. Let (P, x) be a compensation stable payoff configuration which is individually rational. Then for all $i \in N$ and $S, T \in P$ with $i \in T$ and $S \neq T$:

$$v(S) \geq v(S \cup \{i\}) - x_i - \max\Big\{0, x(T \setminus \{i\}) - v(T \setminus \{i\})\Big\}.$$

This implies that for all $i \in N$ and $S, T \in P$ with $i \in T$ and $S \neq T$

$$x_i \geq v(S \cup \{i\}) - v(S) \quad \text{or} \quad x(T \setminus \{i\}) > v(T \setminus \{i\}).$$

Since (P, x) is individually rational $x_i \geq 0 = v(\{i\}) - v(\{\emptyset\})$. We conclude that for all $i \in N$ and $S, T \in P \cup \emptyset$ with $i \in T$ and $S \neq T$, it holds that

$$x_i \geq v(S \cup \{i\}) - v(S) \quad \text{or} \quad x(T \setminus \{i\}) > v(T \setminus \{i\}).$$

So, (P, x) is contractually stable. ■

Note that the proof of Proposition 3.3 in fact implies that any pull-in stable payoff configuration which is individually rational is contractually stable.

We have seen that individually stable outcomes may not always exist. However, if they do, they are necessarily individually rational, as stated in Proposition 3.4.

Proposition 3.4 *Any individually stable payoff configuration is individually rational.*

A trivial restriction on the characteristic value function that ensures the existence of individual stable outcomes is a restriction that ensures the existence of individually rational payoff configurations of which the grand coalition is an element.

Proposition 3.5 *Let* (N, v) *be a coalitional game such that* $v(N) \geq \sum_{i \in N} v(\{i\})$. *Then the payoff configuration* $(\{N\}, x)$ *with* $x_i \geq v(\{i\})$ *for every player* $i \in N$ *is an individually stable outcome of* (N, v).

On the other hand, contractual and compensation stable outcomes, can be found in any coalitional game. However, not all of these outcomes may be individually rational.

4 Mutual Insurance

We apply our stability concepts to the formation of mutual insurance groups by producers who may experience an idiosyncratic negative shock. As shocks are idiosyncratic producers may smoothen production by insuring each other. Moreover, producers self-organize in groups and thus they form mutual insurance groups in which they may share the value of the group, which is given by the total output produced by the members. To illustrate this setting, we adopt the model of an insurance market studied by Rothschild and Stiglitz (1976). Here we will refer to a coalition as an *insurance group*. We use the three types of stability concepts to analyze payoff configurations which differ in terms of the risk composition of the insurance group. In particular we discuss the pooling and separating outcomes which have received much attention in the literature. In Rothschild and Stiglitz (1976) framework, the separating type of outcomes is the only type that may be stable.

First, we describe the demand for insurance against negative production shock. There is a finite set of players N. All players are expected output maximizers each with the same increasing and strictly concave production function Y defined over an amount of input a, *i.e.*, $Y : \mathbb{R} \to \mathbb{R}$ with $Y' > 0$ and $Y'' < 0$. Each player is endorsed with an initial amount of input stock w. With some probability π a player incurs a damage of his stock which has a equivalent of D where $D < w$. There are two groups of players, L and H, forming a partition of N. Players in L have a low probability π_L of incurring a damage. Those in H have a high probability π_H. So $\pi_L < \pi_H$.

We assume that each insurance group offers the same amount of insurance to all members equalling the total damage, *i.e.*, in case her stock is damaged, a producer will receive the same amount of not damaged stock of input as the amount of her damaged stock. The group may require, however, a different contribution "fee" per unit of insurance measured in terms of stock. The contribution fee $q(S)$ of group S is determined by a break-even condition given by:

$$|S|q(S)D - \pi_L|S \cap L|D - \pi_H|S \cap H|D = 0.$$

Hence the contribution fee charged by a group S is

$$q(S) = \frac{|S \cap L|}{|S|}\pi_L + \frac{|S \cap H|}{|S|}\pi_H. \tag{1}$$

So, the contribution fee depends only on the relative size of the risk-pool of the insurance group. In particular, $q(S) \in [\pi_L, \pi_H]$. It is lowest when an insurance group consists of low-risk players only, and it is highest when it consists of high-risk players only.

The value of an insurance group S is defined to be the total production of its members.

Formally,

$$v_\pi(S) = |S|\, Y\Big(w - q(S)D\Big) \quad \text{for all } S \in 2^N \setminus \{\emptyset\}. \tag{2}$$

A coalitional game (N, v_π) derived from the above described tuple $(L, H, Y, w, D, \pi_L, \pi_H, q)$ with $N = L \cup H$ and v_π defined by (2) is called *a mutual insurance coalitional game* to which we refer as an *insurance game* for brevity. Note that the value function is not zero-normalized. However this affects neither the definitions of stability nor the results in Section 3.

The next example is used to illustrate the mutual insurance setting.

Example 4.1 Consider $L = \{1, 2\}$ and $H = \{3\}$, so $N = \{1, 2, 3\}$. The probabilities of incurring a damage are given by $\pi_L = 0.1$ and $\pi_H = 0.7$, respectively. Every player is endowed with the same amount of stock $w = 10$ while $D = 9$. Every player has production abilities represented by the same increasing and strictly concave production function Y defined by $Y(a) = \sqrt{a}$ for $a \geq 0$.

The break-even contribution fee of each insurance group is calculated using Equation (1). The numbers are given below:

$$q(S) = \begin{cases} 0.1 & : \quad S = \{1\}, \{2\}, \{1, 2\}; \\ 0.3 & : \quad S = \{1, 2, 3\}; \\ 0.4 & : \quad S = \{1, 3\}, \{2, 3\}; \\ 0.7 & : \quad S = \{3\}. \end{cases}$$

Using the definition of group value given by Equation (2) and the break-even contribution fees, we obtain the following value function: $v_\pi(\{1\}) = v_\pi(\{2\}) = 3$, $v_\pi(\{3\}) = 1.9$, $v_\pi(\{1, 2\}) = 6$, $v_\pi(\{1, 3\}) = v_\pi(\{2, 3\}) = 5$, and $v_\pi(N) = 8.1$.

In this insurance game all individually stable payoff configurations (P, x) are of the form $P = \{N\}, x(N) = 8.1$ and $x_1 \geq 3$, $x_2 \geq 3$, $x_3 \geq 1.9$.

The contractual and compensation stable payoff configurations (P, x) coincide and are given by $P = \{N\}$ and $x(N) = 8.1$. ♦

In the rest of the section the analysis is focused on pooling[8] and separating types of payoff

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[8] Unlike Rothschild and Stiglitz (1976) we refer to *pooling* payoff configuration only as those outcomes which

configurations. Below we give the formal definitions of these outcomes.

Definition 4.2 *Let* (N, v_π) *be an insurance game. A payoff configuration is called* ***pooling*** *if* $P = \{N\}$.

Definition 4.3 *Let* (N, v_π) *with* $N = L \cup H$ *be an insurance game. A payoff configuration* (P, x) *is called* ***separating*** *if* $P = \{L, H\}$.

We first show that the value function of an insurance game satisfies superadditivity.

Lemma 4.4 *Let* (N, v_π) *be an insurance game. Then for all* $S,T \in 2^N$ *with* $S \cap T = \emptyset$

$$v_\pi(S) + v_\pi(T) \quad \leq \quad v_\pi(S \cup T).$$

Proof. Let (N, v_π) be an insurance game derived from $(L, H, Y, w, D, \pi_L, \pi_H, q)$ with $N = L \cup H$ and v_π defined by Equation (2). Without loss of generality, consider $S, T \in 2^N \setminus \{\emptyset\}$ with $S \cap T = \emptyset$. Then

$$\begin{aligned}
& v_\pi(S) + v_\pi(T) = \\
& \quad |S| Y\left(w - \frac{|S \cap L|\pi_L + |S \cap H|\pi_H}{|S|} D\right) + |T| Y\left(w - \frac{|T \cap L|\pi_L + |T \cap H|\pi_H}{|T|} D\right) \\
= & \quad |S \cup T| \left\{ \frac{|S|}{|S \cup T|} Y\left(w - \frac{|S \cap L|\pi_L + |S \cap H|\pi_H}{|S|} D\right)\right. \\
& \quad \left. + \frac{|T|}{|S \cup T|} Y\left(w - \frac{|T \cap L|\pi_L + |T \cap H|\pi_H}{|T|} D\right)\right\} \\
\leq & \quad |S \cup T| \left\{ Y\left(w - \frac{|S \cap L|\pi_L + |S \cap H|\pi_H + |T \cap L|\pi_L + |T \cap H|\pi_H}{|S \cup T|} D\right)\right\} \\
= & \quad v_\pi(S \cup T).
\end{aligned}$$

The above inequality follows from the strict concavity of Y. The inequality holds as equality in three cases: $S, T \subset L$; $S, T \subset H$; or $|S| = |T|$ with $q(S) = q(T)$. ■

Theorem 4.5 *In an insurance game we have*

(i) Any pooling payoff configuration is compensation stable;

(ii) Any pooling payoff configuration is contractually stable;

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contain the grand coalition.

(iii) Any individually rational pooling payoff configuration is individually stable;

(iv) An individually rational pooling payoff configuration exists.

Proof. Lemma 4.4 implies that any pooling payoff configuration is a payoff configuration of maximum social worth. Therefore, *(i)* and *(ii)* follow from Theorem 3.2. *(iii)* is immediate from the definitions of individual rationality and individual stability: the only coalition to which a player may deviate from the grand coalition is the empty set, such deviation cannot lead to a higher payoff in an individually rational pooling outcome, while *(iv)* follows from the fact that $v_\pi(N) \geq \sum_{i \in N}(v_\pi(\{i\}) - v(\emptyset))$, which is a consequence of Lemma 4.4. ■

Next, we discuss the stability of the separating payoff configurations.

Theorem 4.6 *In an insurance game we have*

(i) No separating payoff configuration is individually stable;

(ii) No separating payoff configuration is compensation stable;

(iii) No individually rational separating payoff configuration is contractually stable.

Proof. Let (N, v_π) be an insurance game derived from $(L, H, Y, w, D, \pi_L, \pi_H, q)$ with $N = L \cup H$ and v_π defined by Equation (2).

(i) Consider a separating payoff configuration $(\{L, H\}, x)$. We will show that the value of the high-risk insurance group is insufficient to give each member at least her outside option of joining the low-risk insurance group. For all $i \in H$, the outside option of joining the low-risk insurance group yields $v_\pi(L \cup \{i\}) - v_\pi(L)$. So for all $i \in H$

$$\begin{aligned}
&v_\pi(H) - |H|\Big(v_\pi(L \cup \{i\}) - v_\pi(L)\Big) = \\
&\qquad |H|Y(w - \pi_H D) - |H|\left((|L|+1)Y\Big(w - \frac{|L|\pi_L + \pi_H}{|L|+1}D\Big) - |L|Y(w - \pi_L D)\right) \\
&= \quad |H|(|L|+1)\left\{\frac{1}{|L|+1}Y(w - \pi_H D) + \frac{|L|}{|L|+1}Y(w - \pi_L D) - Y\Big(w - \frac{|L|\pi_L + \pi_H}{|L|+1}D\Big)\right\} \\
&< \quad 0.
\end{aligned}$$

Here, the inequality follows from the strict concavity of Y.

(ii) Consider a separating payoff configuration $(\{L, H\}, x)$. We will first show that total coalition structure value increases when a high-risk player joins the low-risk group.

For any $i \in H$

$$\begin{aligned} & v_\pi(L \cup \{i\}) + v_\pi(H \setminus \{i\}) - \Big(v_\pi(L) + v_\pi(H)\Big) \\ = \; & (|L|+1)Y\Big(w - \frac{|L|\pi_L + \pi_H}{|L|+1}D\Big) - Y(w - \pi_H D) - |L|Y(w - \pi_L D) \\ = \; & (|L|+1)\left\{Y\Big(w - \frac{|L|\pi_L + \pi_H}{|L|+1}D\Big) - \frac{1}{|L|+1}Y(w - \pi_H D) - \frac{|L|}{|L|+1}Y(w - \pi_L D)\right\} \\ > \; & 0. \end{aligned}$$

The above inequality follows from the strict concavity of Y.

Using the efficiency of the payoffs we have

$$v_\pi(L \cup \{i\}) + v_\pi(H \setminus \{i\}) - \Big(v_\pi(L) + x_i + x(H \setminus \{i\})\Big) > 0.$$

The above inequality implies that at least one of the inequalities below holds for any $i \in H$

$$\Big(v_\pi(L \cup \{i\}) - v_\pi(L) - x_i\Big) > 0 \quad \text{or} \quad v_\pi(H \setminus \{i\}) - x(H \setminus \{i\}) > 0.$$

So for any $i \in H$,

$$\begin{aligned} v_\pi(L) \; & < \; v_\pi(L \cup \{i\}) - x_i - \max\Big\{0, x(H \setminus \{i\}) - v_\pi(H \setminus \{i\})\Big\} \\ \text{or} & \\ x(H \setminus \{i\}) \; & < \; v_\pi(H \setminus \{i\}) - \max\Big\{0, x_i - \Big(v_\pi(L \cup \{i\}) - v_\pi(L)\Big)\Big\}. \end{aligned}$$

Therefore, at least one of the pull-in and push-out conditions is violated.

(iii) Consider an individually rational separating payoff configuration $(\{L, H\}, x)$. By individual rationality, for all $i \in H$

$$x_i \geq v_\pi(\{i\}) = Y(w - \pi_H D).$$

By the efficiency of the payoff vector, for all $i \in H$

$$x(H) = v_\pi(H) = |H|Y(w - \pi_H D).$$

Combining both conditions, we obtain that $x_i = Y(w - \pi_H D)$ for all $i \in H$.

Hence, if any player wants to leave the high risk-group, the rest of the members will grant permission: for all $i \in H$

$$x(H \setminus \{i\}) = |H - 1| Y(w - \pi_H D) = v_\pi(H \setminus \{i\}).$$

To show that this outcome is not contractually stable, we need to show that the value of the high-risk group is insufficient to give all of its members their best outside option, *i.e.*, what they can get by joining the low-risk insurance group. This is already shown in the proof of *(i)* above. ■

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