Local Dynamic Core and Its Applications

Arnold Polanski [1]

Abstract

We consider cooperative games that unfold along histories. Each coalition that cooperates after a history creates a history-dependent surplus,executes intra-coalitional transfers and induces a continuation subgame. The recursive application of the core concept to this dynamic setting leads to the definition of the local dynamic core (LDC). We construct LDC allocations for some representative games and show that the LDC is time-consistent. Furthermore, we propose a non-cooperative implementation of the LDC, which is based on the core implementation game by Moldovanu and Winter (Moldovanu, B., Winter, E., 1995, Order independent equilibria, Games and Econ. Behavior 9, 21-34).

Key words: Dynamic Games; Cooperative Games; Core Implementation

[1] Queen’s University Management School, Queen’s University Belfast, 25 University Square, Belfast BT71NN, tel. +44 28 9097 5024, e-mail: a.polanski@qub.ac.uk

1 Introduction

Human activity often unfolds in a series of interdependent group actions such that a cooperation by a group of individuals influences future utility-possibility sets of other groups. For example, a successful collaboration in a research team enables firms, which rely on this research, to be more productive in the future. Similarly, the departure of a seller-buyer pair from a two-sided market after transaction, changes the trade possibilities of remaining participants. In many cases, members of a group share the cost of cooperation expecting that it will induce further actions and future benefits. For instance, a citizens' initiative can start a non-profit project hoping to elicit further contributions from the public. If the widespread support results in the successful completion of the project, its initiators are able to consume the jointly created public good.

In this work, we model multi-period cooperations as local cooperative dynamic (LCD) games. Local refers to the fact that genuine cooperation takes place in "local coalitions" like buyer-seller pairs or firms. Larger coalitions will not be ruled out but they will have no means to impose any course of action upon their productive subcoalitions. Dynamic means that a game unfolds along a history. The latter consists of a (possibly empty) sequence of local coalitions that have cooperated in the past. After any history, there is a collection of coalitions, which are eligible for cooperation. If an active local coalition cooperates, it creates a (negative) surplus, executes intra-coalitional utility transfers and induces a continuation subgame. In particular, externalities can arise when the induced subgame benefits or harms other coalitions. Noteworthy, decentralized and sequential cooperation lies at the heart of LCD games as opposed to the classic superadditive games, which are static and often implicitly centralized.

In a static game, the core allocation refers to the condition where no coalition of players can improve by gathering and redistributing the resources among its members. It is not obvious, however, how to extend this principle to a dynamic framework. One of the issues that arise here is the possibility of time inconsistency of the core: Coalitions may find it desirable to change their initial plans, given their endowments at some future date. We shall be interested in a cooperative equivalent to the subgame perfect equilibrium (SPE) in the non-cooperative game theory, i.e., in an equilibrium concept that is self-enforcing along any path of the game. In non-cooperative games, the SPE ensures the optimal behavior of the player at the root node of a subtree. A similar purpose accomplishes the local dynamic core (LDC) by prescribing that after any history, each active coalition is allocated at least the value that it could obtain by cooperating and, subsequently, receiving LDC imputations in the ensuing subgame. On the other hand, an LDC allocation must be feasible after each history of the game: There must exist a path of the game that is compatible with the value allocated in the LDC.

After discussing related literature in Section 2 and presenting LCD games and LDC allocation in sections 3 and 4, we compute in Section 5 LDC allocations for instances of games that have been studied in the literature. In particular, we analyse an LCD version of the two-sided homogeneous good market with an exogenous connection structure and extend the analysis by endogenizing the network creation. Furthermore, we study the provision of public goods by groups and the (re)selling of information goods. As a further contribution, we propose in Section 6 a non-cooperative implementation of the LDC by adapting the core implementation game by Moldovanu and Winter (2005).

2 Related Literature

Time inconsistency has attracted the interest of economists for many years. Gale (1978) and Becker and Chakrabarti (1995) explore this issue in the Arrow-Debreu models with dated commodities when agents distrust the forward contracts signed at the first date. Gale introduces the sequential core which consists of allocations that can not be improved upon by anyone at any date, by a suitable reallocation of available coalitional resources. He argues that the sequential core might be empty. The institution of money can, however, act as a substitute for trust and secure the existence of sequential core. In a similar model with an infinite horizon and capital goods, Becker and Chakrabarti (1995) hypothesize that the capital accumulation can provide a nonmonetary trust mechanism without the necessity of introducing a monetary institution. They propose recursive core as an allocation such that no coalition can improve upon its consumption stream at any time, given its accumulation of assets up to that period. They show, in particular, that for every allocation of consumption in the initial core, one can find a distribution of capital stocks among the agents such that no coalition will renege at any date on the initial core contracts.

By contrast to the deterministic models by Gale (1978) and Becker and Chakrabarti (1995), Predtetchinski et al. (2002, 2006) adapt the classical core to dynamic situations involving uncertainty. In a two-stage exchange economy with unknown state of the nature in period one, they define the strong sequential core as the set of allocations that can not be improved upon by any coalition before and after the state is revealed. If only credible deviations are considered, i.e., deviations which cannot be improved upon by any subcoalition of the deviating coalition, the allocation is said to be in the weak sequential core.

In the exchange economy with uncertainty, considered by Dutta and Vohra (2005) and Wilson (1978), agents known their private information and have some probability assessment over the information of others. The critical issue that arises in defining an appropriate core notion is the specification of information that agents in a coalition are allowed to use in constructing objections. Wilson developed two distinct approaches that deal with this issue: The coarse core which is based on the assumption that a coalition can focus its potential objection on an event if and only if the event is common knowledge within the coalition and the fine core which is based on the idea that the act of forming a coalition allows all members of the coalition to decide how much of their private information they wish to share with each other. The credible core in Dutta and Vohra (2005) lies between the fine core and the coarse core and is based on the idea that a coalition can coordinate its potential objection over an event that can be credibly inferred from the nature of the objection being contemplated.

There have also been some attempts to adapt the core to dynamic setting beyond the framework of exchange markets. For finite sequences of exogenously specified TU-games with the same set of players, Kranich et al. (2005) construct dynamic equivalents to the core solution. The strong sequential core allows the coalitions to deviate at any stage of the game, but done so in one period, they must also deviate in all remaining stages. By contrast, in the weak sequential core only credible deviations are considered, i.e., deviations which cannot be blocked by any subcoalition. Filar and Petrosjan (2000) enrich this approach by considering dynamic cooperative games in which the characteristic function evolves over time in accordance with a difference or differential equation. They introduce also mechanisms by which the players may analyse the game with respect to the goal of attaining time consistency. The latter problem is analyzed further in the context of cooperative differential games by Petrosjan (2005).

The present work departs in essential ways from the existing literature. It abandons the specific framework of exchange economies and also the linear evolution of cooperative games. The salient feature of the present model is the path-dependence of cooperation. Group actions create value or entail cost that depends on the "state of the world" in which they are carried out. On the other hand, once executed, the actions change the existing state by ushering a new environment which determines the productivity of further cooperations. Perhaps closest in spirit to our work is the concept of recursive core [1] allocation in partition function form games with trans-

[1] different from the namesake concept in Becker and Chakrabarti (1995).

ferable utilities (e.g., Huang, and Sjöström, 2006). The latter allocation is computed recursively: Each coalition $T \subseteq N$ obtains the maximum value which is consistent with a recursive core allocation in $N \backslash T$. This construction resembles the recursiveness of our definition of the LDC in Section 3. An important difference between games in partition function form and LCD games lies in the fact that in the former the players are removed from the society, once they have cooperated, while in the latter they can participate in several coalitions along a history. Consequently, the respective solution concepts are not directly related.

3 Local Cooperative Dynamic Games and the Local Dynamic Core

The focus of this work is on multistage group interactions, in which the outcome of a "local" cooperation by a group at each stage depends on past group cooperations. More precisely, we define LCD games with transferable utility as evolving along a history. A history $h = \{S_1, ..., S_L\}$, $S_i \subseteq N$, of length L consists of a sequence of coalitions from the set of players N, which have cooperated in the past. A history does not relate directly to the notion of time: The time spans needed for coalitions to cooperate may vary. Also, periods of cooperation may be separated by prolonged inactivity. What matters is the actual sequence of coalitions as it uniquely defines the state of the world which, in turn, determines the productivity of each group of players. The set of all histories is denoted by H. It contains, in particular, the empty history $h^{\varnothing} = \{\}$ and, in some games, infinite histories $h^{\infty} = \{S_1, S_2, ...\}$. After any $h \in H$, there is a collection $a(h)$ of active coalitions that possess a technology to create the value (or incur the investment cost) $v^S(h)$, transfer utility within the coalition and induce the history $h \cup S$. In particular, negative values $v^S(h)$ can be interpreted as an investment that enhances future cooperation. The condition on admissible coalitions, incorporated in $a(h)$, imposes certain restrictions on the set H that rule out incompatible sequences of events,

$$h \in H \Rightarrow h \cup S \in H \Leftrightarrow S \in a(h).$$

In summary, a LCD game $\Phi(h)$ with transferable utility, which starts after the (empty) history $h \in H$, is a collection,

$$\Phi(h) = \{N, a(h'), v(h')\}_{h \subseteq h' \in H},$$

where $v(h') := \{v^S(h')\}_{S \in a(h')}$. We shall also say that the history $h \in H$ is terminal if no productive cooperation is possible thereafter. This happens either because $a(h) = \varnothing$ or because $v^S(h') = 0$ for all $h' \supseteq h$ and $S \in a(h')$. We call a game finite if any non-terminal history has a finite length (as measured by the number of cooperating coalitions) and infinite otherwise. Furthermore, we will call a subgame of $\Phi(h)$ any LCD game $\Phi(h')$ that starts after $h' \supseteq h$.

An integral part of a local cooperation after a history h is an agreement on utility transfers among the members of an acting coalition $S \in a(h)$ that are carried out before the history $h \cup S$ arrives. A vector of utilities $d \in R^{\#S}$ such that $d_S(h) = v^S(h)$ will be called a spot imputation (allocation) for the active coalition S.[2]

An important feature of our model is the lack of the superadditivity assumption. Unlike enforceable agreements on one-time cooperation and transfers within an active coalition (i.e., spot imputations), intertemporal or inter-coalitional agreements are not binding: The players trust neither the promises of future payments for present actions nor the promises of future actions for present payments. Therefore, a super-coalition that is composed of active local coalitions cannot enforce any course of actions agreed by the latter.

Finally, we assume that utility is additive. The total utility (imputation) of player i, along the history h, is the sum of all spot imputations that i receives in coalitions, to which i belongs, and that cooperate along h. Although there is no explicit discounting of utilities, it can be easily incorporated via the family of functions $v(h')_{h' \subseteq h}$, which decrease with the length of the history h'.

For an LCD game $\Phi(h)$, we define the local dynamic core (LDC) recursively,

Definition 1 *The total imputation $d(h) \in R^N_+$ is in the LDC of the finite game $\Phi(h)$ whenever,*

$$
\begin{aligned}
&(DO) \quad \forall S \in a(h), \quad d_S(h) \geq v^S(h) + d_S(h \cup S), \\
&(DF) \quad d(h) > 0 \Rightarrow \exists S \in a(h): \\
&\qquad d_S(h) = v^S(h) + d_S(h \cup S), \quad d_k(h) = d_k(h \cup S),\ k \notin S.
\end{aligned} \tag{1}
$$

The dynamic optimality condition (DO) guarantees that every active coalition $S \in a(h)$ obtains at least the utility $v^S(h)$ of cooperation plus an LDC imputation $d_S(h \cup S)$ of the post-cooperation subgame $\Phi(h \cup S)$. The dynamic feasibility condition (DF) defines the path, along which the LDC imputations are generated. We shall call such a path an LDC or - less formally - an equilibrium history. Furthermore, the condition (DF) stipulates that players outside of the acting coalition S do not

[2] We use the notation $d_S = \sum_{i \in S} d_i$ and $\#S$ = cardinality of the set S.

receive spot imputations. A player $k \notin S \in a(h)$ expects after h her continuation imputation $d_k(h \cup S)$ that is an element of an LDC imputation in the continuation subgame $\Phi(h \cup S)$. Note that an LDC specifies implicitly all spot imputations along an LDC path. Furthermore, the recursive computation of the LDC is closed as for every terminal node (history) $\widetilde{h}$ it holds that $d(\widetilde{h}) = 0$ and all non-terminal histories are finite.

There may be many LDC histories that induce the same LDC allocation. On the other hand, LDC allocations may fail to exist for some games (i.e., the set of equilibrium histories may be empty). Demandingly, the existence of the LDC imputation for the game $\Phi(h)$ requires that such imputations exist for all subgames $\Phi(h')$, $h' \supseteq h$. Obviously, the non-existence problem can be mitigated if we replace the condition (DO) by a weaker one, based on Shapley's and Shubik's (1966) ϵ-core,

$$(DO^{\epsilon}) \qquad \forall S \in a(h), \quad d_S(h) \geq v^S(h) + d_S(h \cup S) - \epsilon.$$

For our purposes, however, more important is the time-consistency of an LDC allocation. By construction, a spot imputation, that is implied by an LDC allocation, will not be blocked by an active coalition in any subgame of the LCD game.

4 LCD Game Tree

As mentioned in the introduction, there is a certain analogy between non-cooperative games in extensive form and LCD games as both can be represented as trees. The analogy is, however, deceptive to some extent. In case of LCD games, a node of the tree corresponds to a set of coalitions, active after a history, while branches are interpreted as coalitions which actually cooperate after histories. By contrast, in non-cooperative games, each node corresponds to a single player, who chooses an action, while branches represent possible actions. There are also differences between our "subgame perfect" solution concept for LCD games and the classical subgame perfect equilibrium (SPE). At each node of the game, the former prescribes an allocation, which satisfy the optimality and feasibility conditions (DO)-(DF), while the latter specifies an optimal action of the corresponding player.

In order to clarify the construction of an LCD game tree and to illustrate the construction of an LDC allocation let us consider the following situation. There are three players, each of whom possessing a piece of exclusive information. Players meet bilaterally and exchange their information, i.e., after a meeting both parties have the same, combined knowledge. The first couple that gathers together all three pieces of information can use the combined knowledge to produce one unit of surplus (say, a marketable good). Hence, the information cannot be transformed into surplus until a pair "knows everything". This situation is described as a cooperative game through the characteristic function,

$$v(\{i\}) = v(\{i,k\}) = 0, \quad i,k \in \{1,2,3\}, i \neq k, \quad v\{\{1,2,3\}) = 1. \qquad (2)$$

Note that this formalization misses the dynamic character of the interaction. Figure 1 below represents an LCD game tree and an LDC allocation that correspond to the situation in the example.

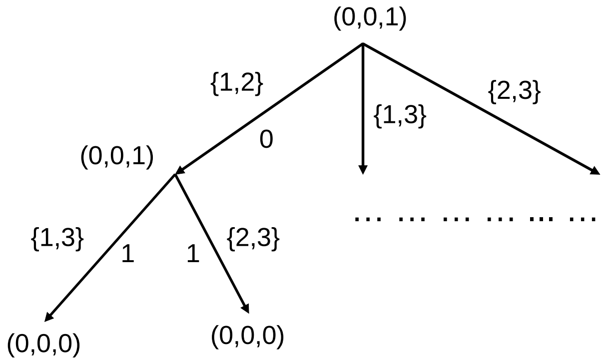

figure 1: An LCD game tree with an LDC allocation.

At the root of the game tree (after the empty history $h^{\varnothing}$), there are three active coalitions, $a(h^{\varnothing}) = \{\{1,2\}, \{1,3\}, \{2,3\}\}$, that all produce zero surplus, $v^{\{i,j\}}(h^{\varnothing}) = 0$. Following the branch (history) $h_{12} := \{\{1,2\}\}$, interpreted as the exchange of information between agents 1 and 2, we arrive at the node where these players have the same, combined knowledge. At this node, each active coalition in the set $a(h_{12}) = \{\{1,3\}, \{2,3\}\}$ acquires, upon meeting, the complete information and can produce the unit surplus, $v^{\{1,3\}}(h_{12}) = v^{\{2,3\}}(h_{12}) = 1$. The histories $\{\{1,2\}$, $\{1,3\}\}$ and $\{\{1,2\}$, $\{2,3\}\}$ are, then, terminal as no surplus can be produced after them. Due to the symmetry of the interaction, we can construct the other branches of the tree in the same manner.

The vectors attached to the nodes in the LCD tree represent an LDC imputation. Starting at the bottom of the tree (after terminal histories), the vector $(0,0,0)$ is the unique LDC imputation. Moving up to the node following the history h_{12}, the single core imputation turns out to be $d(h_{12}) = (0,0,1)$. This is due to the fact that, after h_{12}, players 1 and 2 have both the same (combined) knowledge while player 3 has the critical piece of information. The latter player can, then, appropriate the entire surplus by cooperating, for instance, with player 1,

$$
\begin{aligned}
(DO) \quad d_{\{1,3\}}(h_{12}) &\geq v^{\{1,3\}}(h_{12}) = 1, \quad d_{\{2,3\}}(h_{12}) \geq v^{\{2,3\}}(h_{12}) = 1, \\
(DF) \quad d_{\{1,3\}}(h_{12}) &= v^{\{1,3\}}(h_{12}) = 1, \quad d_{\{2\}} = 0, \\
&\Longrightarrow d(h_{12}) = (0, 0, 1).
\end{aligned}
$$

Analogously, after the histories $h_{13} = \{\{1, 3\}\}$ and $h_{23} = \{\{2, 3\}\}$, the core imputations are $d(h_{13}) = (0, 1, 0)$ and $d(h_{23}) = (1, 0, 0)$, respectively.

Moving up to the root of the LCD game, we realize that the three unit vectors satisfy the LDC conditions at the beginning of the game: No active coalition in $a(\varnothing)$ can block any of these imputations and the surplus allocated in each of the unit vectors is compatible with an LDC path. For example, the core imputation $d(\varnothing) = (0, 0, 1)$ verifies (DO),

$$
\begin{aligned}
d_{\{1,2\}}(\varnothing) = 0 &\geq v^{\{1,2\}}(\varnothing) + d_{\{1,2\}}(h_{12}) = 0, \\
d_{\{1,3\}}(\varnothing) = 1 &\geq v^{\{1,3\}}(\varnothing) + d_{\{1,3\}}(h_{13}) = 0, \\
d_{\{2,3\}}(\varnothing) = 1 &\geq v^{\{2,3\}}(\varnothing) + d_{\{2,3\}}(h_{23}) = 0,
\end{aligned}
$$

and it is generated along the LDC history $\{\{1, 2\}, \{1, 3\}\}$,

$$
(DF) \quad d_{\{1,2\}}(\varnothing) = v^{\{1,2\}}(\varnothing) + d_{\{1,2\}}(h_{12}) = 0, \quad d_3\{\varnothing\} = d_3(h_{12}) = 1.
$$

Note that as player 3 does not participate in the coalition $\{1, 2\}$ after the empty history, she receives her continuation payoff $d_3(h_{12}) = 1$.

It is not difficult to verify that the three unit vectors and the null vector are the only LDC imputations. On the other hand, the core in the static formulation (2) of the game contains a continuum of allocations that includes the unit vectors but not the null vector. We conclude, therefore, that the LDC can be an effective refinement tool of the static core but it also introduces non-core allocations that are dynamically sustainable.

In the next section, we construct LDC imputations for some representative LCD games. In order to compare these allocations to the classical or static core, we restore the superadditivity in these games. The characteristic function $v^S(h) : H \to R_+$ in the superadditive version of the game $\Phi(h)$ computes the maximum value that a coalition $S \subseteq N$ can achieve, when its subcoalitions join their forces after the history h,

$$
v^S(h) := \max_{T \in a(h), T \subseteq S} \{v^T(h) + v^S(h \cup T)\}, \tag{3}
$$

where $v^S(h) = 0$ for all terminal histories and whenever $v^T(h) + v^S(h \cup T) < 0$

for all $T \in a(h)$. The static core $c(h^{\varnothing}) \in R_{+}^{N}$ is an imputation such that,

$$c_S(h^{\varnothing}) \geq v^S(h^{\varnothing}), \quad \forall S \subseteq N,$$

with equality for $S = N$. The examples in the next section confirm that the relation between the LDC and the static core is not clear-cut. They show also that the static core is prone to time inconsistency in dynamic settings as it may include allocations that can not be supported in the core of any subgame.

5 Applications

5.1 Bilateral Cooperation in Networks

There has been some recent interest in the impact of the connection structure in a two-sided market on the price of an homogeneous good. For example, Corominas-Bosh (2004) and Kranton and Minehart (2001) investigate the price formation in network-restricted markets via a multilateral bargaining game and an auction mechanism, respectively. We recast this problem as an LCD game and find an LDC allocation that reflects the structure of the network. The latter is characterized for our purposes by a classical result from the graph theory. The Gallai-Edmonds (GE hereafter) Structure Theorem partitions the set N of nodes that are connected by the set L of links in an arbitrary (not necessary bipartite) graph (N, L) into three subsets. The subset $D \subseteq N$ contains all vertices which are not covered by at least one maximum matching,[3] $A \subseteq N \backslash D$ is the subset of all vertices, adjacent to at least one node in D and, finally, $C = N \backslash (D \cup A)$. If (N, L) is a bipartite graph (bigraph), the GE partition induces a unique decomposition of the graph (N, L) into three disjoint subgraphs. For example, if $N = S \cup B$ represents B buyers and S sellers embedded in a network, the decomposition yields three types of components, intuitively described as excess demand, excess supply and balanced.

[3] A matching m in the graph (N, L) is a subset of non-intersecting links,

$$m \subseteq L : l_1, l_2 \in m \Rightarrow l_1 \cap l_2 = \varnothing.$$

We say that a vertex v is covered by a matching m if $v \in l \in m$. We say further that a matching m is maximum if it is maximal in cardinality.

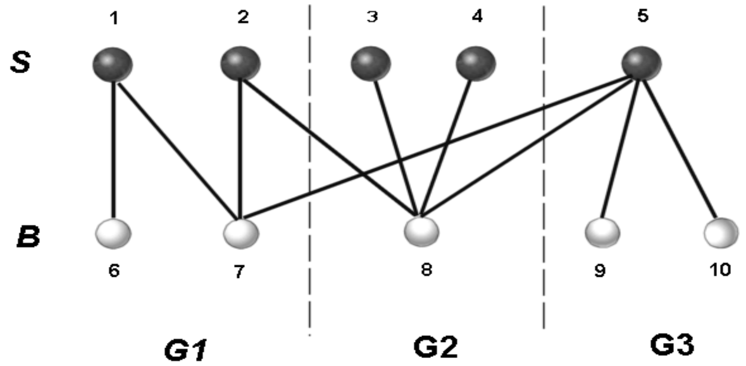

figure 2. GE-decomposition of a bigraph.

The vertices in the upper (lower) row in figure 2 belong to the set S (B) and represent sellers (buyers). The set $S \cup B$ is partitioned into subsets $A= \{8, 5\}$, $D= \{3, 4, 9, 10\}$ and $C = \{1, 2, 6, 7\}$. The GE decomposition yields the balanced subnetwork $G1$ and the excess supply (demand) components $G2$ ($G3$).

We consider an LCD network game on an arbitrary network such that each connected pair of nodes can create and share a normalized surplus of one. After cooperation, the pair is removed from the graph and the involved players do not receive further payoffs. A history h is a sequence of pairs of nodes that have been deleted from the set N after trading. It induces the set of remaining nodes $N(h) \subseteq N$ and the set of remaining links $L(h) \subseteq L$, as specified in the rules of the game,

$$\forall l \in L(h) = a(h), \quad v^l(h) = 1,$$
$$N(h \cup l) = N(h) \backslash l, \quad L(h \cup l) = L(h) \backslash L^l(h),$$
$$N(h^{\varnothing}) = N, \quad L(h^{\varnothing}) = L,$$

where $L^l(h) := \{c \in L(h) : c \cap l \neq \varnothing\}$ is the subset of links that cover any node in the set l after the history h. The next proposition states that along any LDC path, a maximum number of pairs trades and the equilibrium payoffs to the nodes in the sets $A(h)$ and $D(h)$ are unique.

Proposition 1 *Let* $(N(h), L(h))$ *be the graph after the (empty) history* h *and* $A(h) \cup C(h) \cup D(h) = N(h)$ *its GE-partition. Then,* $d(h) \in LDC(h)$ *implies,*

$$i) \quad d(h) \; is \; efficient, \quad ii) \quad d_i(h) = 1 \, (0), \quad \forall i \in A(h) \, (D(h)).$$

Proof: All proofs are relegated to the Appendix.

For the special case of bigraphs, the existence of an LDC allocation can be proven. This result does not extend, however, to all graphs as can be easily verified in the example of a complete graph with three nodes.

Corollary 2 *Let $(N(h), L(h))$ be a bipartite graph after the (empty) history h and $A(h) \cup C(h) \cup D(h) = N(h)$ its GE-partition. Then,*

$$\exists\, d(h) \in LDC(h) : \\ d_a(h) = 1,\ \forall a \in A(h), \quad d_v(h) = 0,\ \forall v \in D(h), \quad d_c(h) = 1/2,\ \forall c \in C(h).$$

Interestingly, the subgame perfect equilibrium of the non-cooperative network game in Corominas-Bosh (2004) coincides with the LDC allocation in the corollary.

As a further step, one could conceive an LCD game which endogenizes the network creation. Specifically, consider a game with S sellers and $B \neq S$ buyers, in which no players are connected at the outset. Players who are not linked and have not traded along the actual history can create a bilateral link at the cost $c \in [0, 1)$. The LDC implies, then, that only $min\{B, S\}$ buyer-seller connections are established and the net surplus $1-c$ in each connection is allocated to the player in the short side of the market. We omit the proof of this result. It parallels the proof of Proposition 1 and is based on the fact that each buyer-seller coalition will claim at least $1 - c$ before the common link is created and 1 afterwards.

Goyal and Vega-Redondo (2004) analyze a model in which all directly or indirectly connected nodes in a network can create value. The payoff to a node i depends on its importance for each cooperating pair j and k. The player i is said to be essential for j and k if i lies on every path that joins j and k in the network (j and k are essential for $\{j, k\}$). If i lies on any path that joins j and k and $i \notin \{j, k\}$, we call it an intermediary. In Goyal and Vega-Redondo (2004), the essential players divide evenly the surplus from cooperation that they facilitate. It can be shown that this payoff structure - and any allocation that splits up the surplus of $\{j, k\}$ among their essential nodes - is a LDC allocation in the LCD version of the game. A history in the latter game consists of a sequence of bilateral cooperations by linked nodes. Interpreting the value of cooperation as derived from the consumption of a good, acquired by k from j via a chain of intermediaries, the intrinsic value of cooperation is zero except for the last transaction which involves the buyer k. The transfers by intermediaries, who cooperate along the equilibrium path, implement the division of k's consumer value among the essential nodes.

The essential conclusions in the above, network-related examples could be obtained by applying the static core. The formalization as LCD games provides, nevertheless, a dynamic bargaining and cooperation framework together with a subgame perfect solution concept. More importantly, the LDC allocation in the first example shows that the efficiency is achieved by local interactions only. This contrasts with the "centralized" optimality brought about by the grand coalition in the static core.

The next two examples rely on the dynamic consistency of the LDC and illustrate its refinement power with respect to the static counterpart.

5.2 Public Goods Produced by Groups

In this subsection, we consider dynamic voluntary contribution games, similar to those discussed e.g. in Admati and Perry (1991), Marx and Matthews (2003) or Lockwood and Thomas (2002). In contrast to the latter models, we assume that the public good is created by sequential contributions of groups and not individuals. There is no mandatory order of contributions nor a prescribed partition of players into groups, but each group must consist of the same number $n > 1$ of members. The public good, valued by all players in one, is produced if and only if $K = N/n > 1$ groups have contributed (we assume that N is a multiple of n). The normalized cost of each group's contribution can be split up arbitrarily among the members. Formally, for all histories h such that there exist players, who have not yet contributed, the active coalitions are defined as,

$$S \in a(h) \Leftrightarrow \#S = n, S \cap T = \varnothing \quad \forall T \in h, \quad v^S(h) = -1. \tag{4}$$

For all histories h such that all players have already contributed, active coalitions consists of players, who have not consumed the public good yet,

$$\{i\} \in a(h) \Leftrightarrow \{i\} \notin h, \quad v^{\{i\}}(h) = 1, \tag{5}$$

Note that (4) and (5) ensure that each player contributes and consumes only once. The imputation $c(h)$ in the static core of the superadditive contribution game must verify only the equality,

$$c_N(h) = v^N(h) = N - K,$$

which leads to a continuum of core allocations. In particular, imputations in which all but one player obtain zero or in which a player bears the total cost in "his" coalition are in the static core. The limited predictive power of the static core contrasts starkly with the LDC allocation in which only the equitable cost division is supported.

Proposition 3 *The unique LDC allocation $d(h)$ in the LCD contribution game verifies,*

$$d_i(h) = 1 - 1/n = (N - K)/N, \quad \forall i \in N.$$

5.3 Information Good Markets with Free Resales

Muto (1986) considers two types of information good (IG) markets with non-positive external effects: Markets with free resales and markets where resales are completely prohibited. He models the trading process as a multilateral bargaining, in which each possessor of information offers simultaneously a price to every demander who can accept or refuse the trade. We rephrase his model as a cooperative game and focus on the case with free resales and no network effects. In the corresponding LCD game, an active coalition consists of a possessor (seller) and a buyer of the information good. A history h is a sequence of pairs that traded the IG. Due to the free resale assumption, the sets of sellers $S(h)$ and buyers $B(h)$ evolve along h. After each transaction, the buyer becomes a future seller of the IG and a competitor to the player from whom he acquired the good. The surplus for each active coalition - which we interpret as the buyer's valuation of the IG - is normalized to one. Since we ignore the depreciation and the cost of reproduction (digital copy), this surplus remains constant after each history. Formally, the LCD game is described as follows,

$$\begin{gathered}\forall S = \{s, b\} : s \in S(h),\, b \in B(h),\, N = S(h) \cup B(h), \quad v^S(h) = 1, \\ S(h \cup \{s, b\}) = S(h) \cup \{b\}, \quad B(h \cup \{s, b\}) = B(h) \backslash \{b\}.\end{gathered} \tag{6}$$

The characteristic function (3) in the superadditive version of the game (6) computes the buyers' surplus in a coalition $T \subseteq N$ that contains at least one possessor of the IG,

$$v^T(h) = \#(T \cap B(h)),$$

if $T \cap S(h) \neq \varnothing$ and $v^T(h) = 0$, otherwise. In one seller markets, say $S(h) = \{s\}$, the static core is compatible with any price structure. In particular, it allows the seller s to extract the full consumer surplus form each buyer. After the first trade, however, s loses the monopoly power and is forced to trade the IG at price zero. This unveils a dynamic inconsistence of some allocations in the static core, which are ruled out by the LDC.

Proposition 4 *In the LCD game (6) the LDC imputation $d(h)$ verifies,*

$$\begin{gathered} i) \quad 1 < \#S(h) \leq N \Rightarrow \\ d_s(h) = 0, \quad \forall s \in S(h), \quad d_b(h) = 1, \quad \forall b \in B(h), \\ ii) \qquad S(h) = \{s\} \Rightarrow \\ d_s(h) + d_b(h) = 1, \quad b \in N \backslash \{s\}, \quad d_k(h) = 1, \quad \forall k \in N \backslash \{s, b\}. \end{gathered} \tag{7}$$

The LDC allocation $d(h)$ in the market with one seller s is supported along the equilibrium path that starts with $h \cup \{s, b\}$, for a particular $b \in N \backslash \{s\}$. The player b is the first buyer of the IG and the only one from whom s can extract some surplus. All other buyers acquire the good at price zero in subgames that follow the trade in $\{s, b\}$. Nevertheless, if also the first buyer obtains the IG for free, any sequence of trades after h supports $d(h)$ and forms an equilibrium path. Hence, the LDC allocation $d_s(h) = 0$, $d_k(h) = 1$ for all $k \in N \backslash \{s\}$, appears to be a focal equilibrium in the LCD game (6) with one seller. In Section 7 we introduce an LDC implementation game and explain how its trembling hand refinement selects this focal equilibrium.

Let us assume now that the IG can be (re)sold only if a connection between a seller and a buyer is established at cost $c \in [0, 1)$. Considering for simplicity the case of at least two sellers and no connections in the market, it holds that the unique LDC allocates the net surplus $1 - c$ of each connection to the connected buyer. This follows from the recursive application of the argument that we put forward for the endogenous network creation in Subsection 5.1.

6 LDC Implementation

Non-cooperative implementation of the core has attracted the interest of economists for many years. Instances of important contributions in this field are Moldovanu and Winter (2005) (MW hereafter), Serrano and Vohra (1997), Kalai et al. (1979) or Perry and Reny (1994). The modified version of the continuos-time model by Perry and Reny (1994) has been used by Huang and Sjöström (2006) to implement the recursive core. In this section we discuss a non-cooperative implementation of the LDC in finite LCD games. Particularly useful in the process is the recursive structure of LDC allocations. For imagine that a dynamic core is SPE implemented in all subgames following the history h, with the induced payoffs $x(h \cup S)$ for all $S \in a(h)$. Then, we can interpret the LCD game which starts after h, as a cooperative TU game with the characteristic function $\widetilde{v}(S) = v^S(h) + x_S(h \cup S)$

for all $S \in a(h)$ and $\tilde{v}(S) = 0$ otherwise. This feature suggests that any non-cooperative implementation of the core, which does not rely on the superadditivity, can be used for LCD implementation.

The present approach follows MW, who analyze a class of non-cooperative games that are based on the NTU game (N, V) and differ from each other by the order of moves. They prove that an allocation is in the core of (N, V) if and only if it is an equilibrium outcome for all games in the class. These games employ a sequential bargaining procedure which the authors describe informally[4] as follows:

"First, consider a function $\varphi : 2^N \rightarrow N$ where $\varphi(S) \in S$ is the first player that has the initiative if the set of players still active in the game is S. An initiator i may shift the initiative to another player, or he may make a proposal. A proposal consists of a coalition S such that $i \in S$, a payoff vector $x^S \in V(S)$, and a responder j, a player of S. The responder can reject or accept the proposal. If the responder rejects, then he becomes the new initiator. If the responder accepts there are two possibilities: If this responder was the last player in S to accept the proposal, then the coalition S forms, it leaves the game, and its members are paid according to x^S. Otherwise, the responder must select the next responder to the existing proposal. After the first coalition has formed, the first player with the initiative is $\varphi(N \backslash S)$, and the game continues in the same fashion. An infinite play results in zero payoffs to all players that remained active in the game."

In our framework, the selection function $\varphi : H \rightarrow N$ chooses after each non-terminal history h the initiator from the set $\cup_{T \in a(h)} T$ of active players. The initiator i proposes a coalition $S \in a(h)$, to which she belongs, a responder[5] $j \in S$ and a vector $x^S \in R^S$ (following the notation in MW, x^S denotes in this section the restriction of $x \in R^N$ to the members of $S \subseteq N$). The proposed vector must satisfy the feasibility condition $x^S_S \leq v^S(h)$. It can include negative entries which are interpreted as contributions of the pertinent players to the cost $v^S(h)$ or as transfers to other members of S. The voting process unfolds as outlined in MW. If the coalition S forms, its members receive or pay x^S and participate in subgames that ensue after $h \cup S$. We shall denote this modified game by $G_\varphi(\Phi(h))$.

MW propose a SPE concept which remains an equilibrium and leads to the same payoff for every possible order of moves. For the game $G_\varphi(\Phi(h))$ the definition reads,

[4] For a formal description, we refer the interested reader to the original work Moldovanu and Winter (2005).

[5] In one-member coalitions, the initiator proposes herself as responder.

Definition 2 *A strategy profile σ is an order independent equilibrium (OIE) for games of the form $G_\varphi(\Phi(h))$ if it satisfies for any order mechanism φ that:*

(1) σ is subgame perfect Nash equilibrium in $G_\varphi(\Phi(h))$.

(2) If σ is played, the payoff in $G_\varphi(\Phi(h))$ is given by a vector $x = x(\sigma, G_\varphi(\Phi(h)))$, independent of φ.

It turns out that an allocation is in the LDC of the underlying game $\Phi(h)$ if and only if it is an outcome of an stationary OIE in $G_\varphi(\Phi(h))$. The next proposition follows from the implementation result in MW.

Proposition 5 *Let $\Phi(h)$ be a finite LCD game and let σ be an OIE in pure, stationary strategies for games of the form $G_\varphi(\Phi(h))$. Then,*

(A) The payoff vector $x = x(\sigma, G_\varphi(\Phi(h)))$ must be in the core of $\Phi(h)$.

(B) For every $d \in LDC(h)$ there exists an OIE σ in pure, stationary strategies that leads to the payoffs d in every game of the from $G_\varphi(\Phi(h))$.

MW's approach differs from the usual concept of implementation as it is founded on a class of games, each for every order of moves, and not a single game. The authors remark, however, that it has a certain appeal because it implements equilibria that are robust to the order of moves in the negotiation process.

If the set of OIE equilibria and the set of LDC allocations are not empty, they will rarely happen to be singletons. To tackle the equilibrium selection problem, we resort to the well-known technique of trembling-hand perfection. First, we perturb the game $G_\varphi(\Phi(h))$ by requiring that the probability for each initiator of submitting a proposal and the probability for each responder of accepting an offer, are not degenerated. For the sake of simplicity, we do not impose any trembles on the proposal itself. Then, the OIE equilibrium σ is (trembling-hand) perfect if and only if it is the limit of a sequence of strategies in the perturbed game and δ_i is the best response to every element of the sequence for all $i \in N$.

In the example of the IG market with one seller (Subsection 5.3), we claim that only one equilibrium survives the trembling-hand refinement procedure. In the selected equilibrium all buyers obtain the IG for free. To see this, suppose for the sake of contradiction that a buyer b proposes or accepts in equilibrium a positive price. Such a strategy cannot form part of a perfect equilibrium because for any element in the converging sequence of strategies, b can profitably deviate by passing the initiative to another buyer. The latter player buys the good with positive probability in which case b obtains the IG for free in the continuation subgame.

7 Extensions

In many situations, groups of individuals are faced with alternative actions and each action induces a different value and history. For instance, in the context of public goods, a coalition may face two possibilities: It can make a specific, cumulative contribution to an existing public good or consume this good immediately precluding further improvements. Similarly, two players may consider various levels of investment in the quality of the common link before extracting profits from the connection. Fortunately, the LDC concept can be readily adapted to the new circumstances. If $\sigma^S(h)$ is the (possibly empty) set of actions, available to a coalition $S \subseteq N$ after the history h, the LDC optimality and feasibility conditions read, respectively,

$$\begin{array}{rl} DO') & \forall S, \sigma \in \sigma^S(h), \quad d_S(h) \geq v^\sigma(h) + d_S(h \cup \sigma), \\ DF') & d(h) > 0 \Rightarrow \exists S, \sigma \in \sigma^S(h) : \\ & d_S(h) = v^\sigma(h) + d_S(h \cup \sigma), \quad d_k(h) = d_k(h \cup \sigma),\ k \notin S. \end{array} \tag{8}$$

Note that now the history is not a sequence of coalitions but of actions taken by coalitions. Similarly, the value $v^\sigma(h)$ of cooperation depends on the history, the coalition and coalition's action. Concerning the implementation of the dynamic core (8) in the game in Section 6, proposals must now include a coalition, a vector of shares, a responder and also an action. All other rules of the game remain unchanged. Provided that the set $\sigma^S(h)$ is compact for every coalition S and history h, the arguments in the proof in Appendix apply and the core (8) can be implemented as the set of order independent equilibria.

8 Appendix

Proof. Proposition 1

$i)$ If $d(h) \in LDC(h)$ then the condition (DO) in the definition (1) implies that $d_v(h) + d_z(h) \geq 1$ for any linked pair $(v, z) \in L(h)$. In particular, in a maximum matching m,

$$\forall (v,z) \in m, \quad d_v(h) + d_z(h) \geq 1$$
$$\Rightarrow \sum_{(v,z)\in m}(d_v(h) + d_z(h)) \geq \#m,$$

where $\#m$ is the cardinality of the maximum matching in $(N(h), L(h))$. Since $\#m$ is the maximum surplus in $(N(h), L(h))$, we have that $d_N(h) = \#m$, i.e., $d_N(h)$ is efficient.

$ii)$ To show $d_v(h) = 0$, $v \in D(h)$, we make use of the fact that for any node $v \in D(h)$ there is a maximum matching m which does not cover v. Then, as shown in part $i)$, the definition (1) implies,

$$\sum_{(w,z)\in m}(d_w(h) + d_z(h)) = \#m = d_N(h) \Rightarrow d_v(h) = 0 \quad if \quad v \notin m.$$

This in turn implies $d_a(h) = 1$ for each node $a \in A(h)$ because, according to a property of GE-partition, any maximum matching m matches nodes in $A(h)$ to nodes in $D(h)$. ■

Proof. Corollary 2

According to the properties of the GE-partition, a link in a bipartite graph $(N(h), L(h))$ never connects a node in the set $D(h)$ to another node in $D(h)$ or to a node in $C(h)$. Given the LDC allocation in the corollary, the condition (DO) in definition (1) is then fulfilled. The properties of the GE-partition ensure also that we can always remove a linked pair from $N(h)$ without changing the GE-partition of the remaining, connected nodes. It is thus possible to construct an LDC path along which the nodes obtain the payoffs as in the corollary. ■

Proof. Proposition 3

The proof is by contradiction. Suppose that $x(h) \neq d(h)$ is in the $LDC(h)$. Note that $x_N(h) \leq N - K$ as $N - K$ is the maximum total payoff that is feasible in the game. Relabel the players by sorting $x(h)$ in ascending order, $x_1(h) \leq ... \leq x_N(h)$. Now, the amount $x_T(h)$, $T = \{1, ..., n\}$, is the sum of $n < N$ smallest imputations.

As the next step we prove that $x_T(h) = n-1 = (N-K)/K$. The sum $x_T(h)$ can not be strictly greater than $(N-K)/K$ because this would imply $x_N(h) \geq Kx_T(h) > N-K$ as $x(h)$ is sorted in ascending order. On the other hand, $x_T(h) < (N-K)/K$ would violate the condition (DO) in the definition (1) along the path, where T is the first cooperating coalition,

$$x_T(h) \geq v^T(h) + x_T(h \cup T) = -1 + N/K = (N-K)/K. \tag{9}$$

The first equality in (9) holds if $x_T(h \cup T) = N/K$ which happens when all subsequent coalitions contribute and the players in T consume the good thereafter. The optimality of subsequent contributions can be shown by a backward induction argument: The last coalition will contribute because its net surplus of cooperation is $n - 1 > 0$, and so will do all preceding coalitions.

Finally, note that each player in T will obtain the same imputation $(N - K)/N$. Otherwise, $x_k(h) > (N - K)/N$ for all $k \geq n$ and hence $x_N(h) > Kx_T = N - K$. On the other hand, it is obvious that the allocation $d(h)$ can be dynamically supported along any path when coalitions split up the contribution cost evenly. ■

Proof. Proposition 4

$i)$ The proof is by induction on the number of buyers. If there is only one buyer b and at least two sellers after the history h, the condition (DO) in the definition (1) implies $d_b(h) = 1$ and zero payoffs to all sellers. Assume now that the claim is valid for $\#B(h) = B > 1$ buyers and $N - B > 2$ sellers and consider the situation with $\#B(h) = B + 1$ buyers. A positive price p that a buyer $b \in B(h)$ pays after h and along an LDC path to a seller $s \in S(h)$ results in $d_s(h) = p$, $d_b(h) = 1 - p$ and $d_{s'}(h) = 0$, $\forall s' \in S(h)\backslash s$ due to the inductive hypothesis and the condition (DF) in (1). But this violates the condition (DO) as $d_{s'}(h) + d_b(h) = 1 - p < 1$. Therefore, all buyer-seller pairs must trade the IG at price zero and any sequence of trades forms an LDC path which induces the imputations in $i)$.

$ii)$ Follows immediately from part $i)$ and the definition (1). The LDC path starts with $h \cup \{s, b\}$ and is continued by any sequence of trading pairs. ■

Proof. Proposition 5

(A) Since the game $\Phi(h)$ is finite we can use induction: We assume that (A) holds for all histories $h \cup S$, $S \in a(h)$ (it holds obviously for a terminal history) and the induced core payoffs are $d(h \cup S)$. For the sake of contradiction, assume that $x \notin LDC(h)$. This happens when at least one of the conditions (DO) or (DF) in (1) is violated. If (DO) is violated there is a coalition $S \in a(h)$ such that $x_S < v^S(h) + d_S(h \cup S)$. Let then $i \in S$ and let $\overline{\varphi}$ be a mechanism with $\overline{\varphi}(h) = i$. We choose $\delta > 0$ and define the improving proposal $(S, y^S) : y_j^S := x_j - d_j(h \cup S) + \delta$ for each $j \in S$ and $y_S^S \leq v^S(h)$. If (S, y^S) is accepted, each player $j \in S$ obtains y_j^S and, in addition, $d_j(h \cup S)$ in the ensuing subgame $G_\varphi(\Phi(h \cup S))$. Hence, the total payoff to j is $x_j + \delta$ and to S amounts to $x_S + \delta(\#S) \leq v^S(h) + d_S(h \cup S)$. More formally, the repetition of the arguments in the proof of Proposition A in [14] shows that the following deviation from σ benefits the player i:

If i is the initiator, and if the history is still h, i proposes (S, y^S); otherwise i follows the strategy σ_i. All other players follow σ_{-i}.

Therefore, if σ is an OIE then the condition (DO) holds, i.e., $x_S \geq v^S(h)+d_S(h\cup S)$ for each active coalition $S \in a(h)$. On the other hand, the proposal (S, e^S), accepted in equilibrium δ by all members of S, verifies by definition $e^S_S \leq v^S(h)$ and leads in the subgame $G_\varphi(\Phi(h \cup S))$ to payoffs $d(h \cup S)$, due to the inductive hypothesis. Hence, the members of S obtain in total,

$$x_S = e^S_S + d_S(h \cup S) \leq v^S(h) + d_S(h \cup S),$$

which together with the condition (DO) implies $x_S = v^S(h) + d_S(h \cup S)$. Furthermore, each player $k \notin S$ is excluded from the offer e^S but obtains in the subgame $G_\varphi(\Phi(h \cup S))$ - again by the inductive hypothesis - the payoff $d_k(h \cup S)$. Thus, k's payoff in the game $G_\varphi(\Phi(h))$, given that S agrees after h in equilibrium σ, is $x_k = d_k(h \cup S)$.

If the OIE outcome $x > 0$ then, for any order φ, there must be at least one coalition $S^\varphi \in a(h)$ which agrees in OIE σ (otherwise the payoffs would be zero to all players). This implies then that the condition (DF) is met and,

$$x_{S^\varphi} = v^{S^\varphi}(h) + d_{S^\varphi}(h \cup S^\varphi), \quad x_k = d_k(h \cup S^\varphi), \quad k \notin S^\varphi.$$

(B)

Since $d \in LDC(h)$ then, by definition (1),

$$d_T \geq v^T(h) + d_T(h \cup T), \quad \forall T \in a(h), \tag{10}$$

$$d > 0 \Rightarrow \exists S \in a(h):$$

$$d_S = v^S(h) + d_S(h \cup S), \quad x_j = d_j(h \cup S), \quad j \notin S. \tag{11}$$

Consider now the following strategy profile σ:

(1) If $i \in S \in a(h)$ is the initiator and the history is still h, then i proposes (S, e^S), where $e^S_k := d_k - d_k(h \cup S)$, $\forall k \in S$. Note that $e^S_S = v^S(h)$ by (11).

(1') If $j \notin S$ is the initiator and the history is still h, then j passes the initiative to $i \in S$.

(2) If j is a responder when the history is still h, the proposal is (T, y^T), $j \in T \in a(h)$, and the set of players that already accepted that proposal is $R \subseteq T$, then $j \in T \backslash R$ accepts if and only if,

$$y^T_k + d_k(h \cup T) \geq d_k,$$

for every $k \in T \backslash R$.

This strategy profile leads obviously to the payoffs d. It remains to show that σ is an equilibrium.

(1-1'): A proposer $i \in S$ can not improve by passing the initiative because it will be returned to her. Furthermore, no proposer j can improve by submitting a valid proposal (T, y^T), $j \in T \in a(h)$, $y^T_T \leq v^T(h)$. If (T, y^T) is accepted, the players in $T \in a(h)$ obtain the total payoffs,

$$y^T_T + d_T(h \cup T) \leq v^T(h) + d_T(h \cup T) \leq d_T,$$

where the last inequality follows from (10). Hence, j's improving proposal (T, y^T) : $y^T_j + d_j(h \cup T) > d_j$ would lead to $y^T_k + d_k(h \cup T) < d_k$ for at least one $k \in T$ and would be rejected.

(2): If $y^T_k + d_k(h \cup T) \geq d_k$ for all $k \in T \backslash R$ then it is optimal for $j \in T \backslash R$ to accept the proposal (T, y^T). Because all other players in $T \backslash R$ follow σ this proposal will be accepted and the payoff to j will be at least d_j. By refusing, j becomes initiator, and we have already seen that his payoff can not be greater than d_j. Finally, if $y^T_k + d_k(h \cup T) < d_k$ for a $k \in T \backslash R$, then the offer (T, y^T) will be rejected by k and j cannot benefit from accepting this offer. ■

References

[1] Admati, A.R., Perry, M., 1991. Joint Projects without Commitment. *Rev. Econ. Studies* 58, 259-276.

[2] Becker, A., Chakrabarti, S.K., 1995. The Recursive Core. *Econometrica* 63, 401-423.

[3] Corominas-Bosch, M., 2004. Bargaining in a Network of Buyers and Sellers. *J. Econ. Theory* 115, 35-77.

[4] Dutta, B., Vohra, R., 2005. Incomplete information, credibility and the core. *Math. Soc. Sciences* 50, 148-165.

[5] Filar, J.A., Petrosjan, L.A., 2000. Dynamic Cooperation Games. *Int. Game Theory. Rev*. 2, 47–65.

[6] Gale, D., 1978. The Core of a Monetary Economy without Trust. *J. Econ. Theory* 19, 456-491.

[7] Goyal, S., Vega-Redondo, F., 2007. Structural holes in social networks. *J. Econ. Theory* 137, 460-492.

[8] Huang, C.Y., Sjöström, J.T., 2006. Non-cooperative Implementation of the Recursive Core. *J. Math. Economics* 42, 771-793.

[9] Kalai, E., Postlewaite, A., Roberts, J., 1979. A group incentive compatible mechanism yielding core allocation. *J. Econ. Theory* 20, 13-22.

[10] Kranich, L., Perea, A., Peters, H., 2005. Core concepts for dynamic TU games. *Int. Game Theory. Rev*. 7, 43–61.

[11] Kranton, R., Minehart, M., 2001. A Theory of Buyer-Seller Networks. *Am. Econ. Rev.* 91, 485-508.

[12] Lockwood, B., Thomas, J.P., 2002. Gradualism and Irreversibility. *Rev. Econ. Studies* 69, 339-356.

[13] Marx, L. M, Matthews, S.A., 2000. Dynamic Voluntary Contribution to a Public Project. *Rev. Econ. Studies* 67, 327-358.

[14] Moldovanu, B., Winter, E., 1995. Order independent equilibria. *Games and Econ. Behav*. 9, 21-34.

[15] Muto, S., 1986. An Information Good Market with Symmetric Externalities. *Econometrica* 54, 295-312.

[16] Perry, M., Reny, P.J., 1994. A Noncooperative view of coalition formation and the core. *Econometrica* 62, 795-817.

[17] Petrosjan L.A., 2005. Cooperative Differential Games in Nowak, A. S., Szajowski, K. (eds.). Advances in Dynamic Games: Applications to Economics, Finance, Optimization and Stochastic Control, Birkhaeuser Boston.

[18] Predtetchinski, A., Herings, P.J.J., Peters, H., 2002. The Strong Sequential Core for Two-Period Economies. *J. Math. Economics* 38, 465-482.

[19] Predtetchinski, A., Herings, P.J.J., Perea, A., 2006. The Weak Sequential Core for Two-Period Economies. *Int. J. Game Theory* 34, 55-65.

[20] Serrano, R., Vohra R., 1997. Non-cooperative implementation of the core. *Soc. Choice and Welfare* 14, 513-525.

[21] Shapley, L. S., Shubik, M., 1966. Quasi-cores in a monetary economy with non-convex preferences, *Econometrica* 34: 805–827.

[22] Wilson, R., 1978. Information, efficiency and the core of an economy. *Econometrica* 46, 807-816.