A Tale of Two Platforms: Dealer Intermediation in the European Sovereign Bond Market

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Appendix A: Value Functions

Proposition 1.

We derive the linear form of the value functions for each of the three inventory states $s = -1, 0, 1$. For this purpose we conjecture that the optimal standardized B2C quotes $(a(s), b(s)) = (\widehat{a}(s) - x_t, \widehat{b}(s) - x_t)$ are independent from the variable x_t. In proposition 2, we show that this is indeed the case under optimal quote setting. Intuitively, dealers earn a cash flow from intertemporal demand intermediation in the B2C market. The expected cash flow created from the customer relationship should therefore not depend on the price level of the asset under consideration. Hence, the value function cannot depend on the process x_t if the dealer starts from a zero inventory level. We therefore impose the condition $V(0, x_t) = V(0) = V$ for all levels of x_t.

For a positive or negative inventory level, however, the value function generally depends on the level of the asset price because the inventory itself is valuable. Next we determine the functional form of $V(1, x_t)$. The case of $V(-1, x_t)$ is analogous. Recall that the stochastic process x_t has binomial innovations $\Delta x_{t+1} \in \{+\epsilon, -\epsilon\}$ of constant and equal probability $\frac{1}{2}$. We further assume that dealers earn (pay) interest on the nominal value $rx_t = \frac{1-\beta}{\beta} x_t$ of their positive (negative) inventory. The transition probabilities follow from Assumption 1 as

$$\begin{array}{lllll}
p_{12} & = & qF^b(R^b - x_{t+1} \leq \widehat{b}(1) - x_{t+1}) & = & q\left(1 + b\left(1\right)d - d\Delta x_{t+1}\right) \\
p_{11} & = & 1 - p_{12} - p_{10} & & \\
p_{10} & = & qF^a\left(R^a - x_{t+1} \geq \widehat{a}(1) - x_{t+1}\right) & = & q\left(1 - a(1)d + d\Delta x_{t+1}\right) \\
p_{01} & = & qF^b(R^b - x_{t+1} \leq \widehat{b}(0) - x_{t+1}) & = & q\left(1 + b\left(0\right)d - d\Delta x_{t+1}\right) \\
p_{00} & = & 1 - p_{01} - p_{0-1} & & \\
p_{0-1} & = & qF^a\left(R^a - x_{t+1} \geq \widehat{a}(0) - x_{t+1}\right) & = & q\left(1 - a(0)d + d\Delta x_{t+1}\right) \\
p_{-10} & = & qF^b(R^b - x_{t+1} \leq \widehat{b}(-1) - x_{t+1}) & = & q\left(1 + b\left(-1\right)d - d\Delta x_{t+1}\right) \\
p_{-1-1} & = & 1 - p_{-10} - p_{-1-2} & = & \\
p_{-1-2} & = & qF^a\left(R^a - x_{t+1} \geq \widehat{a}(-1) - x_{t+1}\right) & = & q\left(1 - a(-1)d + d\Delta x_{t+1}\right)
\end{array} \quad . \tag{1}$$

Using the transition probabilities, we express the value functions as

$$\begin{aligned}
V(1, x_t) \;=\; & \frac{1}{2}\beta\left[V(1, x_t + \epsilon)(1 - p^+_{10}) + [B - b(1)]\, p^+_{12} + V(0, x_t + \epsilon)p^+_{10} + [a(1) + x_t]\, p^+_{10}\right] + \\
& + \frac{1}{2}\beta\left[V(1, x_t - \epsilon)(1 - p^-_{10}) + [B - b(1) - c]\, p^-_{12} + V(0, x_t - \epsilon)p^-_{10} + [a(1) + x_t]\, p^-_{10}\right] \\
& + \beta r x_t,
\end{aligned} \tag{2}$$

where $p^+_{s_1 s_2}$ and $p^-_{s_1 s_2}$ denotes the transition probability from inventory state s_1 to s_2 for innovations $\Delta x_{t+1} = +\epsilon$ and $\Delta x_{t+1} = -\epsilon$, respectively. Inspection of equation (2) shows that repeated substitution for the terms $V(1, x_t + \epsilon)$ and $V(1, x_t - \epsilon)$ yields a sequence of discounted terms $\beta^i x_t$ (with $i = 1, 2, 3...$) and a sequence of constants $V(0)$,

B, $b(1)$ and $a(1)$ all independent of x_t. A similar consideration follows from the development of

$$\begin{aligned} V(-1,x_t) &= \frac{1}{2}\beta\left[V(-1,x_t+\epsilon)(1-p^+_{-10})+\left[a(-1)-A\right]p^+_{-1-2}+V(0,x_t+\epsilon)p^+_{-10}+\left[b(-1)+x_t\right]p^+_{-10}\right]+ \\ &+\frac{1}{2}\beta\left[V(1,x_t-\epsilon)(1-p^-_{-10})+\left[a(-1)-A\right]p^-_{-1-2}+V(0,x_t-\epsilon)p^-_{-10}+\left[b(-1)+x_t\right]p^-_{-10}\right] \\ &-\beta r x_t \end{aligned}$$

Again sequential substitution gives discounted terms only in $\beta^i x_t$ (with $i = 1, 2, 3...$) and a sequence of constants. Under the usual transversality condition that this sequence has an upper bound, there exist some constant k_x for which the value function can be expressed as

$$\begin{aligned} V(1,x_t) &= V(1)+k_x x_t \\ V(-1,x_t) &= V(-1)-k_x x_t \end{aligned},$$

for the inventory levels 1 and -1, respectively. Next we show that $k_x = 1$. Using

$$\begin{aligned} &\frac{1}{2}\left[V(1,x_t+\epsilon)(1-p^+_{10})+V(1,x_t-\epsilon)(1-p^-_{10})\right] \\ =& \frac{1}{2}V(1,x_t+\epsilon)(1-q(1+d\epsilon-da(1))+\frac{1}{2}V(1,x_t-\epsilon)(1-q(1-d\epsilon-da(1) \\ =& V(1,x_t)(1-q(1-da(1)))-k_x q d\epsilon^2 = V(1,x_t)(1-\mathcal{E}_t\left(p_{10}\right))-k_x q d\epsilon^2 \end{aligned}$$

and

$$\frac{1}{2}\left[V(0,x_t+\epsilon)p^+_{10}+V(0,x_t-\epsilon)p^-_{10}\right] = V(0,x_t)q(1-da(1)) = V(0,x_t)p_{10},$$

we rewrite the value function as

$$\begin{aligned} V(1,x_t) &= \beta V(1,x_t)(1-p_{10})-\beta k_x q d\epsilon^2+\beta V(0,x_t)p_{10}+\beta\left[B-b(1)\right]p_{12}+\beta\left[a(1)+x_t\right]p_{10}+\beta r x_t \\ &= \beta V(1,0)(1-p_{10})-\beta k_x q d\epsilon^2+\beta V(0,0)p_{10}+\beta\left[B-b(1)\right]p_{12}+\beta a(1)p_{10}+ \\ &+\beta k_x x_t(1-p_{10})+\beta x_t p_{10}+(1-\beta)x_t. \end{aligned}$$

A comparison of coefficients with $V(1, x_t) = V(1) + k_x x_t$ implies that $k_x = \beta k_x(1 - p_{10}) + \beta p_{10} + 1 - \beta$ or $k_x = 1$. The value function for the inventory $s = 1$ is therefore given by $V(1, x_t) = V(1) + x_t$. An analogous argument applies to the inventory $s = -1$ where we find also find $k_x = 1$. Defining the concavity parameter $\nabla = V(0) - V(1)$ implies the linear form in proposition 1.

Appendix B: Optimal B2C Quotes

Proposition 2.

(i) The dealer value function (equation 1 in the paper) can be expanded as

$$\begin{aligned} V(1,x_t) &= \max_{\{a(s),b(s)\}} \beta\mathcal{E}_t \left[\begin{array}{c} [V(1) + x_{t+1} + B - b(1)]\,p_{12} + [V(1) + x_{t+1}]\,p_{11} + \\ + [V(0) + a(1) + x_t]\,p_{10} + r x_t \end{array} \right] \\ V(0,x_t) &= \max_{\{a(s),b(s)\}} \beta\mathcal{E}_t \left[\begin{array}{c} [V(1) + x_{t+1} - b(0) - x_t]\,p_{01} + V(0)\,p_{00} + \\ + [V(-1) - x_{t+1} + a(0) + x_t]\,p_{0-1} \end{array} \right] \\ V(-1,x_t) &= \max_{\{a(s),b(s)\}} \beta\mathcal{E}_t \left[\begin{array}{c} [V(0) - b(-1) - x_t]\,p_{-10} + [V(-1) - x_{t+1}]\,p_{-1-1} + \\ + [V(-1) - x_{t+1} - A + a(-1)]\,p_{-1-2} - r x_t \end{array} \right]. \end{aligned} \tag{3}$$

For each of the three state variables, we find the first order conditions by differentiation with respect to the corresponding quoted B2C prices $a(s)$ and $b(s)$. This implies the 6 first order conditions stated in proposition 2. The second order conditions are trivially fulfilled since the Hessian matrix is $-2d\mathbf{I}_3$ and therefore negative definite.

(ii) It is more difficult to derive the condition on the concavity parameter ∇ which depends on the B2B spread S. From proposition 1, we know that the value function has a linear representation in the state variable x_t. In order to solve for ∇, we can write the value function (3) for optimal B2C quotes as

$$\mathbf{V}(s,x_t) = \beta\mathcal{E}_t \left[\mathbf{M}\mathbf{V}(s,x_{t+1}) + \widetilde{\mathbf{\Lambda}}\right] = \beta\mathbf{M}\mathbf{V}(s,x_t) + \mathbf{\Lambda}_0 + \mathbf{\Lambda}_x x_t + \mathbf{\Phi} \tag{4}$$

where $\mathbf{M}$ denotes the transition matrix and where we define vectors

$$\mathbf{\Lambda}_0 = \beta \left[\begin{array}{c} \left[-\frac{S}{2} - b(1)\right] p_{12} + a(1)p_{10} \\ -b(0)p_{01} + a(0)p_{0-1} \\ -b(-1)p_{-10} + \left[a(-1) - \frac{S}{2}\right] p_{-1-2} \end{array} \right], \tag{5}$$

$$\mathbf{\Lambda}_x = \beta \left[\begin{array}{c} 1+r \\ p_{0-1} - p_{01} \\ -(1+r) \end{array} \right] = \left[\begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right],$$

$$\mathbf{\Phi} = \beta\mathcal{E}_t \left[\begin{array}{c} \Delta x_{t+1}\,(p_{12} + p_{11}) \\ \Delta x_{t+1}\,(p_{01} - p_{0-1}) \\ -\Delta x_{t+1}\,(p_{-1-1} + p_{-1-2}) \end{array} \right] = \beta \left[\begin{array}{c} -qd\mathcal{E}_t\,(\Delta x_{t+1})^2 \\ -2qd\mathcal{E}_t\,(\Delta x_{t+1})^2 \\ -qd\mathcal{E}_t\,(\Delta x_{t+1})^2 \end{array} \right] = \beta q d \epsilon^2 \left[\begin{array}{c} -1 \\ -2 \\ -1 \end{array} \right].$$

Subtracting the vector $\mathbf{\Lambda}_x x_t$ from both sides in equation (4) we obtain

$$\mathbf{V}(s,0) = \mathbf{V}(s) = \beta\mathbf{M}\mathbf{V}(s,0) + \mathbf{\Lambda}_0 + \mathbf{\Phi}.$$

Hence, the concavity parameter $\nabla = V(0) - V(1)$ is implicitly characterized by

$$\mathbf{V}(s)=\begin{bmatrix} V(1)\\ V(0)\\ V(-1)\end{bmatrix}=\begin{bmatrix} V-\nabla\\ V\\ V-\nabla\end{bmatrix}=(\mathbf{I}-\beta\mathbf{M})^{-1}(\Lambda_0+\mathbf{\Phi}). \tag{6}$$

The vector Λ_0 denotes the expected payoffs in each state. It is independent of both the current price process x_t and its innovation Δx_{t+1}. The vector $\mathbf{\Phi}$ captures the state specific adverse selection risk with respect to shocks to the price process x_t. The matrix $\mathbf{M}$ of transition probabilities can be written as

$$\mathbf{M} = \mathcal{E}_t\begin{bmatrix} p_{12}+p_{11} & p_{10} & 0\\ p_{01} & p_{00} & p_{0-1}\\ 0 & p_{-10} & p_{-1-1}+p_{-1-2}\end{bmatrix}= \tag{7}$$

$$= \begin{bmatrix} 1-[q\{1-a(1)d\}] & q\{1-a(1)d\} & 0\\ q\{1+b(0)d\} & 1-q\{1+b(0)d\}-q\{1-a(0)d\} & q\{1-a(0)d\}\\ 0 & q\{1+b(-1)d\} & q\{1+b(-1)d\}\end{bmatrix}. \tag{8}$$

Substituting the relevant elements of (1) into (5) and using (7), we can rewrite

$$(\mathbf{I}-\beta\mathbf{M})\,\mathbf{V}(s)-\Lambda_0-\mathbf{\Phi}=\mathbf{0}$$

or

$$\begin{bmatrix} 8\beta q+\beta d^2 q\left(4\nabla^2+S^2-16\epsilon^2\right)+4d\{2\nabla(2+\beta(q-2))-\beta qS+4V(0)(\beta-1)\}\\ V(0)-\dfrac{\beta q\left\{4d^2\epsilon^2-(d\nabla-1)^2\right\}}{2d(\beta-1)}\\ 8\beta q+\beta d^2 q\left(4\nabla^2+S^2-16\epsilon^2\right)+4d\{2\nabla(2+\beta(q-2))-\beta qS+4V(0)(\beta-1)\}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The second equation can be solved for $V(0)$ in terms of ∇. The first and third equations are identical and we substitute $V(0)$ into either to obtain

$$f_{b2c}\left(\nabla,S,\epsilon^2,q,d\right)=\frac{1}{4}\left(\beta dq\nabla^2\right)+\nabla\left(-1+\beta-\frac{3\beta q}{2}\right)-\frac{\beta q\left\{S(dS-4)+16d\epsilon^2\right\}}{16}=0. \tag{9}$$

This B2C schedule characterizes the inventory concavity parameter ∇ of a dealer's value function under optimal B2C quotes and for any B2B spread S. It is depicted in Figure 2.

Appendix C: Competitive Pricing in the B2B Market

Proposition 3.

To determine the expected loss of liquidity provision in the interdealer market, it is useful to denote by $(n(1),n(0),n(-1))>0$ the number of traders with inventories $1,0$, and -1, respectively. We assume furthermore that the probability q of customer arrival in the B2C market is sufficiently small so that $\frac{1}{2}q<\frac{n(1)}{n(-1)}<\frac{2}{q}$ holds. Liquidity at the best B2B ask price is only demanded by dealers who experience an negative inventory shock from -1 to -2 and are therefore forced to rebalance. The respective probability p_{-1-2} (see equations (1)) is given by $q(1-a(-1)d+d\epsilon)$ when $\Delta x_{t+1}=\epsilon$ (with probability $\frac{1}{2}$) and $q(1-a(-1)d-d\epsilon)$ when $\Delta x_{t+1}=-\epsilon$ (with probability $\frac{1}{2}$). The liquidity supplying dealer (at the ask) experiences an expected loss if the liquidity demand is more likely to occur for $\Delta x_{t+1}=\epsilon$ than $\Delta x_{t+1}=-\epsilon$. If market orders (due to rebalancing needs) in the B2B market were unrelated to the dynamics of Δx_{t+1}, then the expected (adverse selection) loss L^A of liquidity provision at the ask would follow as

$$L^A=\frac{1}{2}\epsilon+\frac{1}{2}(-\epsilon)=0.$$

But since execution probabilities for limit order supplies depend on Δx_{t+1}, we have instead

$$L^A=\text{prob}(\Delta x_{t+1}=\epsilon \mid \text{Execution})\ \epsilon+\text{prob}(\Delta x_{t+1}=-\epsilon \mid \text{Execution})\ (-\epsilon), \tag{10}$$

where $\text{prob}(\Delta x_{t+1}=\epsilon\ |\text{Execution})>\frac{1}{2}$ denotes the probability of $\Delta x_{t+1}=\epsilon$ conditional on execution of the liquidity supply at the ask. Using Bayes rule implies

$$\text{prob}(\Delta x_{t+1} = \epsilon \mid \text{Execution})=\frac{\text{prob}(\Delta x_{t+1}=\epsilon\cap\text{Execution})}{\text{prob}(\Delta x_{t+1}=\epsilon\cap\text{Execution})+\text{prob}(\Delta x_{t+1}=-\epsilon\cap\text{Execution})} \tag{11}$$

$$\text{prob}(\Delta x_{t+1} = -\epsilon \mid \text{Execution})=\frac{\text{prob}(\Delta x_{t+1}=-\epsilon\cap\text{Execution})}{\text{prob}(\Delta x_{t+1}=\epsilon\cap\text{Execution})+\text{prob}(\Delta x_{t+1}=-\epsilon\cap\text{Execution})}. \tag{12}$$

We calculate the expected number of (unit) market order as $n(-1)q(1-a(-1)d\pm d\epsilon)$ (for $\Delta x_{t+1}=\pm\epsilon$, respectively) and the number of (unit) liquidity supplies at the best ask as $n(1)$. The execution probability for each liquidity supplying dealer then follows as

$$\begin{aligned}\text{prob}(\Delta x_{t+1} &= \epsilon\cap\text{Execution})=\frac{1}{2}\frac{n(-1)q(1-a(-1)d+d\epsilon)}{n(1)}\\ \text{prob}(\Delta x_{t+1} &= -\epsilon\cap\text{Execution})=\frac{1}{2}\frac{n(-1)q(1-a(-1)d-d\epsilon)}{n(1)}.\end{aligned}$$

Both expressions are bounded between 0 and 1 for $\frac{1}{2}q<\frac{n(1)}{n(-1)}$. Substitution into equations (11) and (12) implies

$$\begin{aligned}\text{prob}(\Delta x_{t+1} &= \epsilon \mid \text{Execution})=\frac{(1-a(-1)d+d\epsilon)}{2(1-a(-1)d)}=\frac{1}{2}+\frac{d\epsilon}{2(1-a(-1)d)}\\ \text{prob}(\Delta x_{t+1} &= -\epsilon \mid \text{Execution})=\frac{(1-a(-1)d-d\epsilon)}{2(1-a(-1)d)}=\frac{1}{2}-\frac{d\epsilon}{2(1-a(-1)d)}.\end{aligned}$$

The expected loss of B2B liquidity supply at the best ask stated in (10) follows as

$$L^A=\frac{d\epsilon^2}{[1-a(-1)d]}=\frac{\epsilon^2}{\frac{1}{d}-a(-1)}=\frac{\epsilon^2}{\frac{1}{d}-\frac{S}{4}-\frac{1}{2d}}=\frac{2\epsilon^2}{\frac{1}{d}-\frac{S}{2}},$$

and an analogous expression holds for $L^B=L^A=L$.

The equilibrium condition equalizes the adverse selection costs L^A with benefits of a balanced inventory ∇, the transaction revenue $\frac{S}{2}$ and order processing costs τ. If all rents from liquidity disappear under perfect supply, we obtain as the B2B equilibrium condition

$$f_{b2b}\left(\nabla,S,\epsilon^2,q,d\right)=\tau-\frac{S}{2}-\nabla+\frac{4d\epsilon^2}{2-Sd}=0. \tag{13}$$

and (for $S \geq 0$)

$$\begin{array}{lllll} A & = & \max(L - \nabla + \tau, 0) & = & \frac{S}{2} \\ B & = & \min(-L + \nabla - \tau, 0) & = & -\frac{S}{2} \end{array}.$$

Appendix D: Existence and Uniqueness of the Equilibrium

Proposition 4:

First, we show that the two equilibrium schedules (9) and (13) have exactly two intersections in the $(\frac{S}{2}, \nabla)$ space as long as the volatility ϵ^2 of the midprice process x_t is below some treshhold $\overline{\epsilon}^2$. This situation is graphed in Figure 2. Second, we argue that only one of the two equilibria is stable. Third, for high levels of volatility with $\epsilon^2 > \overline{\epsilon}^2$ no equilibrium exists in which both the B2B and B2C market function simultaneously.

To characterize the shape of the B2C equilibrium schedule, we calculate the partial derivatives of the implicit function f_{b2c} giving

$$\begin{bmatrix} \frac{\partial f_{b2c}}{\partial S} \\ \frac{\partial f_{b2c}}{\partial \nabla} \\ \frac{\partial f_{b2c}}{\partial \epsilon^2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{8}\beta q\,(dS - 2) > 0 \\ -1 + \frac{1}{2}\beta\left[2\,(1-q) + qd\left(\nabla - \frac{1}{d}\right)\right] < 0 \\ -\beta dq < 0 \end{bmatrix}. \tag{14}$$

We have $\frac{\partial f_{b2c}}{\partial S} > 0$ because the uniform distribution was restricted to have $\frac{S}{2} < \frac{1}{d}$. Moreover, $\frac{\partial f_{b2c}}{\partial \nabla} < 0$, because $q < 1$ and $\nabla < \frac{1}{d}$. To verify the condition $\nabla < \frac{1}{d}$, take into consideration that the ask quote $a(1) = \frac{1}{2}\left(\frac{1}{d} - \nabla\right) > 0$ in equation (3) in the paper needs to be positive. The B2C schedule has the derivatives

$$\frac{\partial \nabla_{b2c}}{\partial S} = -\frac{\frac{\partial f_{b2c}}{\partial S}}{\frac{\partial f_{b2c}}{\partial \nabla}} = \frac{\frac{1}{8}\beta q\,(dS-2)}{-1 + \frac{1}{2}\beta\left[2\,(1-q) + qd\left(\nabla - \frac{1}{d}\right)\right]} > 0 \quad \text{and} \quad \frac{\partial^2 \nabla_{b2c}}{\partial S^2} < 0.$$

In the $(\frac{S}{2}, \nabla)$ space the B2C schedule is therefore increasing in S with a decreasing slope.

Next, we examine the B2B schedule (13). Its intercept with the vertical axis is found by evaluating equation (13) at $S = 0$, which gives $2d\epsilon^2 + \tau$. The B2B schedule has derivatives

$$\frac{\partial \nabla_{b2b}}{\partial S} = -\frac{1}{2} + \frac{4d^2\epsilon^2}{(2 - Sd)^2} \quad \text{and} \quad \frac{\partial^2 \nabla_{b2b}}{\partial S^2} < 0. \tag{15}$$

At $S = 0$, we find $\frac{\partial \nabla_{b2b}}{\partial S} = -\frac{1}{2} + 2d^2\epsilon^2 < 0$, because the maximum value of ϵ^2 is $\frac{1}{4d^2}$. Equation (13) is quadratic. Its minimum is obtained for $\frac{S}{2} = \frac{1}{d} - \sqrt{2\epsilon^2}$. For $\frac{1}{d} - \sqrt{2\epsilon^2} < \frac{S}{2} < \frac{1}{d}$, the slope is positive. Importantly, $\frac{\partial^2 \nabla_{b2b}}{\partial S^2} > 0$ for the B2B schedule and $\frac{\partial^2 \nabla_{b2c}}{\partial S^2} < 0$ for the B2C schedule implies that both schedules intersect exactly twice as long as the volatility ϵ^2 is not too large. Of the two equilibria Z_L and Z_H shown in Figure 2, only Z_L with lower values of S and ∇ is stable. Deviation of a liquidity supplier in the B2B market to a lower spread S immediately attracts all the market orders from other dealers. The less favorable B2B quotes become irrelevant. The reverse argument does not hold, which demonstrates the stability of equilibrium Z_L. Finally, as ϵ^2 becomes large, the B2B and B2C schedule no longer intersect and no market equilibrium exists. The volatility level ϵ^2 at which both schedule touch in one tangency point characterizes the threshold value $\overline{\epsilon}^2$ for breakdown of the joint equilibrium in both markets.

Appendix E: Approximation to Solution

Proposition 5:

The market equilibrium is characterized by the B2C schedule (9) and B2B schedule (13), respectively. This equilibrium is linearize around the value $\epsilon^2 = 0$. Let $(\frac{\overline{S}}{2}, \overline{\nabla})$ denote the equilibrium values which correspond to $\epsilon^2 = 0$, which fulfill $\overline{\nabla} = \tau - \frac{\overline{S}}{2}$. The first order approximation of the B2B equation (13) follows as

$$\nabla = \tau - \frac{S}{2} + \alpha_{1v}\epsilon^2, \tag{16}$$

with a parameter $\alpha_{1v} = 4/(\frac{2}{d} - \overline{S}) > 0$. The corresponding approximation of the B2C equation (9) follows as

$$0 = \alpha_{2c} + \alpha_{2s}\frac{S}{2} + \alpha_{2\nabla}\nabla + \alpha_{2v}\epsilon^2, \tag{17}$$

with parameters

$$\alpha_{2c} = \tfrac{1}{4}\beta qd\left[\left(\tfrac{\overline{S}}{2}\right)^2 - \left(\overline{\nabla}\right)^2\right] < 0, \qquad \alpha_{2s} = -\tfrac{1}{4}\beta qd\left[\overline{S} - \tfrac{2}{d}\right] > 0,$$
$$\alpha_{2\nabla} = \beta - 1 + \tfrac{1}{2}\beta q\left(d\overline{\nabla} - 3\right) < 0, \qquad \alpha_{2v} = -\beta qd < 0.$$

Substitution of equation (16) into equation (17) implies

$$\frac{S}{2} = -\frac{\alpha_{2c} + \alpha_{2\nabla}\alpha_{1c}}{\alpha_{2s} + \alpha_{2\nabla}\alpha_{1s}} - \frac{\alpha_{2vol} + \alpha_{2\nabla}\alpha_{1v}}{\alpha_{2s} + \alpha_{2\nabla}\alpha_{1s}}\epsilon^2 = \gamma_{4c} + \gamma_{4v}\epsilon^2, \tag{18}$$

where we find for the coefficients

$$\gamma_{4c} = -\frac{\alpha_{2c} + \alpha_{2\nabla}\alpha_{1c}}{\alpha_{2s} + \alpha_{2\nabla}\alpha_{1s}} = \tau - \frac{\alpha_{2c} + \alpha_{2s}\tau}{\alpha_{2s} - \alpha_{2\nabla}} < \tau, \qquad \gamma_{4v} = -\frac{\alpha_{2vol} + \alpha_{2\nabla}\alpha_{1v}}{\alpha_{2s} + \alpha_{2\nabla}\alpha_{1s}} = \alpha_{1v}.$$

The inventory concavity parameter follows (after substitution of equation (18) into equation (16)) as

$$\nabla = \tau - \frac{S}{2} + \alpha_{1v}\epsilon^2 = \tau - \gamma_{4c}, \tag{19}$$

and substitution into the first order conditions implies

$$\begin{bmatrix} a(-1) \\ a(0) \\ a(1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2d} \\ \frac{1}{2d} \\ \frac{1}{2d} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{S}{2} \\ \nabla \\ -\nabla \end{bmatrix} = \begin{bmatrix} \gamma_{1c} \\ \gamma_{2c} \\ \gamma_{3c} \end{bmatrix} + \begin{bmatrix} \gamma_{1v} \\ 0 \\ 0 \end{bmatrix}\epsilon^2,$$

where $\gamma_{1c} = \frac{1}{2d} + \frac{1}{2}\gamma_{4c}$, $\gamma_{2c} = \frac{1}{2d} + \frac{1}{2}(\tau - \gamma_{4c})$, $\gamma_{3c} = \frac{1}{2d} - \frac{1}{2}(\tau - \gamma_{4c})$ and $\gamma_{1v} = \frac{1}{2}\gamma_{4v} < \gamma_{4v}$. It follows directly that $\gamma_{1c} > \gamma_{2c} > \gamma_{3c}$ and $\gamma_{2c} > \gamma_{4c}$. Analogous relationships apply at the bid side.

Corollary 1:

Let $p(s) = E(\frac{n(s)}{n})$ denote the probability distribution of traders over the three inventory states s and assume that it does not depend on the volatility ϵ^2. We define the expected B2C spreads

$$\overline{a} = \sum_{s=-1,0,1} p(s)a(s)g(a(s) \text{ and } \overline{b} = \sum_{s=-1,0,1} p(s))b(s)g(b(s).$$

where $g(a(s))$ and $g(b(s)$ denotes the probability that the respective customer quote is accepted. Furthermore, $g(a(s)) = 1 - a(s)d$ with $0 < a(s) < \frac{1}{d}$. The expected B2C ask price is given by

$$\overline{a} = \sum_{s=-1,0,1} p(s)a(s)g(a(s) = \sum_{s=-1,0,1} p(s)a(s)\left[1 - a(s)d\right] = \sum_{s=-1,0,1} p(s)\left[a(s) - a(s)^2 d\right].$$

The derivative with respect to volatility ϵ^2 follows as

$$\frac{\partial \overline{a}}{\partial \epsilon^2} = \sum_{s=-1,0,1} p(s)\left[1 - 2a(s)d\right]\frac{\partial a(s)}{\partial \epsilon^2} < \frac{\partial a(-1)}{\partial \epsilon^2} = \gamma_{1v} < \gamma_{4v} = \frac{\partial}{\partial \epsilon^2}\frac{S}{2} = \frac{\partial}{\partial \epsilon^2}A.$$

A similar argument applies to the bid side. Hence,

$$\frac{\partial}{\partial \epsilon^2}\left[A - \overline{a}\right] > 0 \quad \text{and} \quad \frac{\partial}{\partial \epsilon^2}\left[\overline{b} - B\right] > 0.$$

Proposition 6.

For the volatility dependence see proposition 5. Combining $a(-1) > a(1)$, $b(-1) > b(1)$ and $\frac{\partial p(-1)}{\partial Imb} < 0$,$\frac{\partial p(1)}{\partial Imb} > 0$, from equation 8 in the paper, implies $\frac{\partial (A-\overline{a})}{\partial Imb} = -\frac{\partial \overline{a}}{\partial Imb} > 0$ and $\frac{\partial (\overline{b}-B)}{\partial Imb} = \frac{\partial \overline{b}}{\partial Imb} < 0$.