Optimal Pricing and Strategic Decisions in Networks

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Abstract

This paper studies optimal pricing and strategic decisions in networks with quadratic objective functions, following Ballester, Calvo-Armengol and Zenou (*Econometrica*, 2006). It focusses on two questions: How do optimal prices and strategies reflect the position of agents in the network? What is the effect of a change in the network structure on optimal pricing and strategic decisions? Using an asymptotic approach, it shows that the ranking of optimal prices and strategies can be reduced to the ranking of simple characteristics of the agent's position in the network. The addition of a link between two agents i and j in the network affects the equilibrium price or strategy of agent k according to the minimum distance between k and i or j in the network. These results are applied to optimal pricing in a network with consumption externalities, and in a model where agents care about the average price observed in their neighborhood.

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1 Introduction

Social networks play an important role in many economic activities. Recognizing this, economists have recently started employing formal analytical tools to study job contact networks (Calvo-Armengol and Jackson (2004)), buyer-seller networks (Kranton and Minehart (2001) and Corominas-Bosch (2004)), risk-sharing networks (Bloch, Genicot and Ray (2007) and Bramoullé and Kranton (2007a)), criminal networks (Calvo-Armengol and Zenou (2004)), networks of strategic alliances (Goyal and Joshi (2001) and Goyal and Moraga Gonzales (2001)), of international trade agreements (Furosawa and Konishi (2005) and Goyal and Joshi (2005)), or of market sharing agreements (Belleflamme and Bloch (2004)). Following the seminal work on network externalities and network industries in the 1980's (Farrell and Saloner (1985) and Katz and Shapiro (1986)), industrial organization theorists have also recently extended models of consumption externalities to allow externalities to depend on an underlying social network (Jullien (2001), Sundarajan (2005) and Banerji and Dutta (2005)).

The economic literature on social networks has for the most part centered around two sets of questions: How do rational, self-interested agents form social networks? How does the structure of the social network affect economic decisions?[1] The second question can itself be subdivided into two questions: For a given network, how is the position of an agent in the network reflected in its economic choices? What are the comparative statics effects of changes in the networks on the agents' decisions?

In general, the structure of networks is so complex that it is hard to obtain answers to these questions. For large networks characterized by their degree distribution, recent work by Galeotti et al. (2006) and Galeotti and Goyal (2007) attempts to answer the questions, by focussing on agents' degrees and deriving relations between the degree of a node and the optimal economic decision. For small networks where agents have quadratic utilities, Ballester, Calvo-Armengol and Zenou (2006) derive a general relation between the optimal decision of an agent, and his Bonacich (1987) centrality in the network.

The paper by Ballester, Calvo-Armengol and Zenou (2006) (from now on BCAZ) serves as the starting point of our analysis. We claim that the general relation they uncover between the weighted Bonacich centrality measure and the agents' optimal strategies is particularly useful when the matrix of interaction is proportional to the matrix representing the social network (or can

[1] For excellent recent surveys on the economic literature on networks, see the books by Sanjeev Goyal (2007) and Matt Jackson (2008).

easily be decomposed as the sum of a matrix of 1s and of the matrix representing the social network, as in their study of criminal networks (Ballester, Calvo-Armengol and Zenou (2004) and Calvo-Armengol and Zenou (2004)). But in many other applications, where the interaction matrix does not possess this simple structure, the weighted Bonacich centrality measure becomes hard to compute, and loses its transparent interpretation. Furthermore, comparative statics effects become hard to sign when the interaction matrix fails to be symmetric.

In response to these difficulties, and in accordance with BCAZ's remark that an interior equilibrium exists if and only if the magnitude of local effects is not too large,[2] we propose in this paper an asymptotic approach to study the effect of social networks on economic decisions when the magnitude of external effects goes to zero. At the outset, we want to stress that, if at first glance the study of social network effects whose magnitude goes to zero may appear to be a self-defeating enterprise, we believe that the intuition gained in the limiting case is very valuable. First, when the matrix of external effects is complex, this may be the only way to handle a problem which would otherwise be intractable. Second, by continuity, the intuition obtained for small external effects continues to hold when the magnitude of externalities increases, so that our qualitative results remain true for a wider range of situations. In order to test this hypothesis, we actually run a sensitivity analysis to check whether our qualitative intuitions carry over to situations with stronger externalities.

Our asymptotic results complement the results of BCAZ, who were the first to show that the vector of optimal decisions can be expressed as a power series of vectors. Using well known results on matrix norms, we first define an upper bound on the approximation obtained when the series is truncated at an arbitrary finite level K. This approximation result is then used to obtain two propositions of practical interest. First, we show that the ranking between two components of the vector of optimal prices or optimal decisions becomes equivalent to the lexicographic ordering of two sequences, which are computed by using the elements in the power series. Hence, it becomes possible to rank the optimal decisions at two nodes as a function of simple characteristics of the nodes, which appear in the first terms of the power series and are (relatively) easy to compute. Second, we show that the effect of the addition of a link between two nodes i and j on the optimal price or decision of node k can be ascertained by computing a single vector. If

[2] This condition on the magnitude of external effects distinguishes BCAZ from another study of optimal public good provision in the network by Bramoullé and Kranton (2007b), who typically find multiplicity of corner equilibria when the external effects are large.

the interaction matrix can be written only as a function of powers of the network matrix, the effect of an additional link between i and j on node k only depends on the minimal distance between k and either i or j.

In the second part of the paper, we apply these asymptotic results to economic models with quadratic objective functions. We first consider a model of consumption externalities where consumers with quadratic-linear utility functions are affected by consumption at nodes which are direct neighbors. If firms compete in quantities, this model is similar to the model of BCAZ, and equilibrium quantities at two nodes depend on the Bonacich centrality. In particular, nodes with higher degree will be served by higher quantities. If firms compete in prices, we again find that prices are higher at nodes with higher degree. However, if firms collude and set the optimal price of a multiproduct monopolist, this result is reversed, and consumers at nodes with higher degree are charged lower prices. This last result is due to the fact that the monopolist realizes that an increase in consumption at nodes with high degrees results in higher demand ar neighboring nodes. We also discuss three other economic applications. In a behavioral model where consumers pay attention to the average price observed in their neighborhood, we find that consumers will be charged higher prices if the sum of the inverse of degrees of nodes in their neighborhood is higher. In BCAZ's model of criminal activities in networks, we rank criminal efforts as a function of the centrality measure of the agents. Finally, we analyze Goyal and Moraga (2001)'s model of R & D efforts in networks of alliances.

The rest of the paper is organized as follows. We introduce the model and derive general asymptotic results in Section 2. Section 3 presents our application to pricing with network externalities. Section 4 is devoted to the other applications. We conclude in the last Section.

2 The Model and General Results

2.1 The Model

2.1.1 Utilities and strategic choices

As in BCAZ, we consider a society of n agents, $i = 1, 2, ...n$, each choosing a strategic variable x_i in $\Re$, with a quadratic objective function:

$$U_i(\mathbf{x}) = \alpha_i x_i + (1 - a_{ii})\frac{1}{2}a_{ii})x_i^2 + \sum_{j \neq i} a_{ij} x_i x_j.$$

The first order optimality condition for agent i yields:

$$x_i = \alpha_i + \sum_{j=1}^{n} a_{ij} x_j.$$

Let $\mathbf{A}$ be the square $n \times n$ matrix, $\mathbf{A} = [a_{ij}]$ and $\mathbf{a}$ the vector $\mathbf{a} = (\alpha_1, ..., \alpha_n)$. Let $\mathbf{I}$ denote the $n \times n$ identity matrix. An interior equilibrium is then defined by the solution to the system of linear equations:

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{a} \tag{1}$$

2.1.2 Matrices, vectors and graphs

Before we analyze the solution to this system of equations, we introduce additional notations on matrices and graphs. We let $\mathbf{U}$ denote the $n \times n$ matrix with all entries equal to 1. We let $\mathbf{G}$ de a zero diagonal matrix with typical term g_{ij}. If $\mathbf{G}$ is symmetric and $g_{ij} \in \{0, 1\}$, the matrix $\mathbf{G}$ represents a *social network*, and we define characteristics of social networks. The *degree* of agent i is the number of direct neighbors of i,

$$\deg i = \sum_j g_{ij}.$$

We let $\mathbf{d}$ denote the vector of degrees of the agents, with typical term $d_i = \deg i$ and $\mathbf{id}$ the vector inverses of degrees with typical term $id_i = \frac{1}{\deg i}$. For any vector $\mathbf{x}$, we will also denote the diagonal matrix formed by using the components of $\mathbf{x}$ by $\Delta(\mathbf{x})$. We let $\mathbf{e}$ denote the vector of 1's. Finally, for any vector $\mathbf{x}$, we denote the projection of the vector on the i axis by $(\mathbf{x})_i$.

Given any graph represented by a matrix $\mathbf{G}$, a *path* is a sequence of nodes, $i^0, i^1, ..., i^m$ such that $g_{i^j i^{j+1}} = 1$ for all $j = 0, ..., m-1$. If $i^0 = i^m$ but no other term appears twice in the sequence, the path is called a *cycle*.

2.1.3 BCAZ's results

BCAZ first observe that the matrix $\mathbf{I} - \mathbf{A}$ can always be decomposed into:

$$\mathbf{I} - \mathbf{A} = \delta\mathbf{I} + \gamma\mathbf{U} - \lambda\mathbf{B},$$

for some triple of positive scalars $(\delta, \gamma, \lambda)$ and some zero diagonal matrix $\mathbf{B}$.[3]Furthermore, dividing both sides of the For the matrix $\mathbf{B}$ and a scalar b, let

[3]As opposed to BCAZ, we are not requiring that $\mathbf{B}$ be nonnegative. Because our results are asymptotic results when local effects become small, we can always assume that the matrix $I - A$ is nonnegative. Hence all properties of nonnegative matrices can be used

$$\mathbf{M}(\mathbf{B}, b) = \sum_{k=0}^{\infty} b^k \mathbf{B}^k.$$

This expression has a simple interpretation when $\mathbf{B} = \mathbf{G}$ is a matrix representing an undirected graph. For any i, j and k, the term g_{ij}^k measures the *total number of paths of length* k *originating at* i *and ending at* j. Two particular measures will be of special interest. We let ν_i^k denote the diagonal elements of G^k, namely the number of paths of length k starting and finishing at i. Notice that, by construction, $\nu_i^1 = 0$, $\nu_i^2 = \deg i$, and $\nu_i^3 = cyc_3 i$, where *cyc*3*i* denotes the number of cycles of length 3 going through node i. We let ϕ_i^k denote the *total number of paths of length* k *originating at* i, $\phi_i^k = \sum_j g_{ij}^k$. The *Bonacich centrality measure* (Bonacich, 1987) of the network represented by the matrix $\mathbf{B}$ with respect to scalar b is defined as:

$$\mathbf{b}(\mathbf{B}, b) = \mathbf{M}(\mathbf{G}, b)\mathbf{e}.$$

If $\mathbf{B} = \mathbf{G}$, the Bonacich centrality measure of player i is the discounted sum of the number of paths originating at i,

$$(\mathbf{b})_i = \sum_{k=0}^{\infty} b^k \phi_i^k.$$

BCAZ's main results can be summarized as follows:

Theorem 1 (BCAZ, Theorem 1) *Let* $\mathbf{B}$ *be symmetric. If the largest eigenvalue of* $\mathbf{B}$, $\mu_1(\mathbf{B})$ *satisfies:* $\mu_1(\mathbf{B}) \leq \frac{\delta}{\lambda}$, *then the system of linear equations (1) has a unique solution* $\mathbf{x}^*$ *given by:*

$$\mathbf{x}^* = \frac{\alpha}{\delta + \gamma \mathbf{b}(\mathbf{B}, \lambda^*)\mathbf{e}} \mathbf{b}(\mathbf{G}, \lambda^*),$$

where $\lambda^* = \frac{\delta}{\lambda}$.

Theorem 2 (BCAZ, Theorem 2) *Consider two games which only differ in the local effects, with* $\mathbf{B} \neq \mathbf{B}'$ *and suppose that* $\mathbf{B}$ *and* $\mathbf{B}'$ *are symmetric, that* $g_{ij} \geq g'_{ij}$, $\mu_1(\mathbf{B}) \leq \frac{\delta}{\lambda}$ *and* $\mu_1(\mathbf{B}) \leq \frac{\delta}{\lambda}$. *Then, the two games admit unique solutions which satisfy:* $\mathbf{x}^*\mathbf{e} \geq \mathbf{x}^{*'}\mathbf{e}$.

(Seneta (1973), p. 40), and in particular the solutions to the system of equations are nonnegative when the vector $\mathbf{a}$ is nonnegative.

These two theorems clearly indicate how a player's position in the network affects his equilibrium choice, and how changes in the network affect the total equilibrium decisions for a broad class of games. However, we argue that, when the initial situation is nonsymmetric, as will occur in some applications, these results have limited practical interest. While the Bonacich centrality measure can be extended to these situations, it loses its direct, transparent interpretation. For nonsymmetric situations, the comparative statics effects of a change in the social network become very difficult to ascertain. These considerations lead us to propose in the next subsection an asymptotic analysis of the system of linear equations which will enable us to obtain simple intuitions in the more complicated applications.

2.2 Asymptotic results on systems of linear equations

In this Section, we derive general asymptotic results on systems of linear equations, which will be used in later Sections to discuss equilibrium strategies in economic models. Recall that we consider a system of n linear equations in n variables given by:

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{a},$$

where $\mathbf{I}$ is the $n \times n$ identity matrix, $\mathbf{A}$ an arbitrary $n \times n$ matrix, $\mathbf{x}$ and $\mathbf{a}$ vectors in $\Re^n$. We will use the l_∞ vector norm, defined by:

$$||\mathbf{A}|| = \max_{i,j} |a_{i,j}|.$$

The following Proposition is a direct application of well-known results on matrix norms.

Proposition 1 *Suppose that $||\mathbf{A}|| < \frac{1}{n}$. Then, the system of linear equations (1) has a unique solution $\mathbf{x}^*$ and*

$$||\mathbf{x}^* - \sum_{k=0}^{K} \mathbf{A}^k \mathbf{a}|| \leq \frac{n^{K+1}||\mathbf{A}||^{K+1}||\mathbf{a}||}{1 - n||\mathbf{A}||}.$$

Proposition 1 shows that when the norm of the matrix $\mathbf{A}$ is small enough, the unique solution of the system of linear equations can be approximated by a polynomial expression in the powers of the matrix $\mathbf{A}$. This approximation will prove particularly useful when the norm of the matrix $||\mathbf{A}||$ becomes arbitrarily small.

Recall that the matrix $\mathbf{A}$ san be decomposed as the sum of idiosyncratic, global and local effects:

$$\mathbf{A} = (1-\delta)\mathbf{I} - \gamma\mathbf{U} + \lambda\mathbf{B}.$$

We now construct sequences of systems of linear equations where the magnitude of local effects goes to zero. While we postulate a specific form of convergence, the model we consider is general enough to encompass all the economic applications studied in later sections. We specifically suppose that the matrix $\mathbf{B}$ and the vector $\mathbf{a}$ can be written as polynomial expressions in β, a positive scalar measuring the magnitude of local effects. Formally, there exist sequences $(\mathbf{B}_1, \mathbf{B}_2, ...)$, $(\mathbf{a}_0, \mathbf{a}_1, ...)$ such that

$$\lambda\mathbf{B} = \sum_{l=1}^{\infty} \lambda^l \beta^l \mathbf{B}_l, \mathbf{a} = \sum_{l=0}^{\infty} \lambda^l \beta^l \mathbf{a}_l \tag{2}$$

whenever the series $\sum_{l=1}^{\infty} \lambda^l \beta^l \mathbf{B}_l$ and $\sum_{l=0}^{\infty} \lambda^l \beta^l \mathbf{a}_l$ are convergent.
We now compute an infinite sequence of vectors c_m which will play a central role in the analysis. We first recall the definition of the *partition* of an integer m.

Definition 1 *A partition of an integer m, $p(m)$ is a sequence of positive integers, $(p_1, ..., p_R)$ such that $\sum_r p_r = m$. The set of all partitions of an integer m is denoted $\mathcal{P}(m)$. By convention, suppose that the partition of 0 is 0 and let $\mathbf{I}$ be the matrix corresponding to $\mathbf{B}_0$.*

Next define the sequence of vectors:

$$c_m = \sum_{t=0}^{m} a_t \frac{\lambda^m}{\delta^{m-t+1}} \Big(\sum_{(p_r)\in\mathcal{P}(m-t)} \prod_r \mathbf{B}_{p_r} \Big). \tag{3}$$

Arguably, this sequence is not easy to compute. However, the first terms are rather straightforward and given by:

$$\begin{aligned}
c_0 &= \frac{1}{\delta}\mathbf{a}_0, \\
c_1 &= \frac{\lambda}{\delta}\mathbf{a}_1 + \frac{\lambda}{\delta^2}\mathbf{B}_1\mathbf{a}_0, \\
c_2 &= \frac{\lambda^2}{\delta}\mathbf{a}_2 + \frac{\lambda^2}{\delta^2}\mathbf{B}_1\mathbf{a}_1 + \frac{\lambda^2}{\delta^3}(\mathbf{B}_1^2 + \mathbf{B}_2)\mathbf{a}_0, \\
c_3 &= \frac{\lambda^3}{\delta}\mathbf{a}_3 + \frac{\lambda^3}{\delta^2}\mathbf{B}_1\mathbf{a}_2 + \frac{\lambda^3}{\delta^3}(\mathbf{B}_1^2 + \mathbf{B}_2)\mathbf{a}_1 + \frac{\lambda^3}{\delta^4}(\mathbf{B}_1^3 + \mathbf{B}_2\mathbf{B}_1 + \mathbf{B}_1\mathbf{B}_2 + \mathbf{B}_3)\mathbf{a}_0.
\end{aligned}$$

Next, define the sequence:

$$\mathbf{f} = (\mathbf{c}_0, \mathbf{c}_1, ..., \mathbf{c}_m, ...).$$

Recall the definition of the lexicographic ordering of sequences:

Definition 2 *The sequence $f = (f_1, ..., f_k, ...)$* lexicographically dominates *the sequence $f' = (f'_1, ..., f'_k)$, $f \prec f'$ if and only if there exists K such that $f_k = f'_k$ for all $k < K$ and $f_K > f'_K$.*

We are now ready to prove our main Theorem, which provides an equivalence between the ranking of the components of the solution $\mathbf{x}^*$ and the lexicographic ordering of the components of the sequence $\mathbf{f}$ when the matrix of local effects $\mathbf{B}$ converges to zero.

Proposition 2 *Consider a sequence of matrices $\{\mathbf{A}_t\}$ defined by equation (2) such that $\mathbf{B}_t$ converges to the zero matrix, with an associated sequence of scalars β_t monotonically converging to 0. There exists $T > 0$ such that, for all $t \geq T$, the system of linear equations (1) has a unique solution x_t^* and*

$$(\mathbf{x}_t^*)_i > (\mathbf{x}_t^*)_j \Leftrightarrow (\mathbf{f}_t)_i \succ (\mathbf{f}_t)_j$$

Proposition 2 relies on a simple intuition. As Proposition 1 shows, the solution of the system of equations can be approximated by a polynomial in β, $\sum_{m=0}^{\infty} \beta^m(\mathbf{c}_m)$. When β converges to 0, the higher powers of the polynomial become negligible, and the first elements of the sequence become sufficient to characterize the solution.[4]

[4]This argument is almost entirely correct. The only point which remains to be checked is that, when β converges to zero, not only do the higher terms of the series vanish, but the series also converges.

Proposition 2 provides a useful tool to rank the components of the vector of solutions to a system of linear equations when the matrix $\mathbf{A}$ converges to zero. Consider for example a system of linear equations where $\mathbf{A} = \lambda\mathbf{G}$ and $\mathbf{a} = \mathbf{e}$. Proposition 2 then states that, at the first order, the ranking between between the equilibrium values for two agents i and j depends on the ranking of the degrees of the agents in the network. If $\lambda > 0$, agents with higher degrees will face higher prices or choose higher equilibrium strategies ; if $\lambda < 0$, agents with higher degree will face lower prices or choose lower equilibrium strategies.

We next consider the effect of changes in the matrix $\mathbf{A}$ on the solutions to the system of linear equations. The following result is immediately obtained, applying arguments which parallel those of Proposition 2.

Proposition 3 *Suppose that, in the decomposition of the matrix* $\mathbf{I} - \mathbf{A}$*,* $\gamma = 0$*. Consider two sequences of matrices and vectors* $\{\mathbf{A}_t, \mathbf{a}_t\}$ *and* $\{\mathbf{A}'_t, \mathbf{a}'_t\}$ *defined by equation (2) , with* $\{\mathbf{B}_t\}$ *and* $\{\mathbf{B}'_t\}$ *converging to the zero matrix, with an associated sequence of scalars* β_t *monotonically converging to* 0*. For any* i*, let* M_i *be the first index for which* $(\mathbf{c}_m)_i \neq (\mathbf{c}'_m)_i$*. Then, there exists* $T > 0$ *such that, for all* $t \geq T$*, both systems of linear equations admit unique solutions* x^*_t *and* $x^{*'}_t$ *and*

$$(x^*_t)_i > (x^{*'} - t) \Leftrightarrow (\mathbf{c}_m)_i > (\mathbf{c}'_m\mathbf{a})_i.$$

Proposition 3 provides a simple criterion to check the effect of a change in the matrix $\mathbf{A}$ on the components of the solution vector in the absence of global effects when the local effects converge to zero. The key to this criterion is to identify the first term in the sequences $\{\mathbf{c}_m\}$ and $\{\mathbf{c}'_m\}$ for which the components $(\mathbf{c}_m)_i$ and $(\mathbf{c}'_m)_i$ differ. While this in general a difficult problem, it has a nice solution when we consider $\mathbf{A} = \lambda\mathbf{G}$, $\mathbf{a} = \mathbf{e}$ and analyze the effect of the addition of one link in the matrix $\mathbf{G}$. Formally, let $\mathbf{G}'$ be obtained from $\mathbf{G}$ by replacing two cells for which $g_{ij} = g_{ji} = 0$ by $g'_{ij} = g'_{ji} = 1$. For any agent $l \neq i, j$, let ζ denote the length of the shortest path between l and either i or j. It is easy to check that $(\mathbf{G}^k\mathbf{e})_l = (\mathbf{G}'^k\mathbf{e})_l$ for all $l < \zeta$ and $(\mathbf{G}^\zeta\mathbf{e})_l < (\mathbf{G}'^\zeta\mathbf{e})_l$. Hence, by Proposition 3, if $\lambda > 0$, the addition of a link between i and j has a positive effect on the price or equilibrium strategy of any other player. If $\lambda < 0$, the addition of a link between i and j on the price or equilibrium strategy of player l will have a positive (respectively negative) effect if the minimal distance between l and either i or j is even (respectively odd).

3 Pricing in a Network

Consider a society where consumers are located along a network with n different nodes. The matrix $\mathbf{G}$, with generic term g_{ij} represents the underlying network. A consumer located at node i only cares about the consumption of consumers located at nodes which are direct neighbors of i. Formally, a consumer located at node i has a quadratic quasi-linear utility function:

$$U_i = q_i - \frac{1}{2}q_i^2 + \lambda \sum_j g_{ij} q_i q_j - p_i q_i. \tag{4}$$

We allow both for $\lambda > 0$ (the case of positive network externalities) and $\lambda < 0$ (the case of negative externalities, or "snob effects"). We suppose that there are n identical firms serving each node of the network. We normalize the constant marginal cost of each firm to zero. We furthermore assume that consumers can only buy from the firm serving their location. Consumer i chooses the quantity q_i in order to maximize her utility, so that the inverse demands are given by

$$p_i = 1 - q_i + \lambda \sum_j g_{ij} q_j. \tag{5}$$

or in matrix form:

$$\mathbf{p} = \mathbf{e} - (\mathbf{I} - \lambda \mathbf{G})\mathbf{q} \tag{6}$$

We consider three different forms of market competition: Cournot competition, Bertrand competition, and perfect collusion where firms act as if a single multiproduct monopolist served all the markets.

3.1 Cournot competition

If the firms choose quantities, each firm will select quantities to maximize profit, resulting in the first-order conditions:

$$1 - 2q_i + \lambda \sum_j g_{ij} q_j = 0. \tag{7}$$

In matrix terms, we have $(\mathbf{I} - \mathbf{A}) = 2I - \lambda \mathbf{G}$ and $a = (1-c)\mathbf{e}$, so that $\delta = 2$, $\mathbf{B}_1 = \mathbf{G}$ and $a_0 = (1-c)\mathbf{e}$. This is a very transparent model, and we can apply BCAZ to show that, if $2 > \lambda \mu_1(\mathbf{G})$, the quantity produced at node i is *increasing in the Bonacich centrality* $\mathbf{B}(\mathbf{G}, \frac{\lambda}{2})$ *of the node if* $\lambda > 0$ and *decreasing in the Bonacich centrality* $\mathbf{B}(\mathbf{G}, \frac{-\lambda}{2})$ *of the node if* $\lambda < 0$.

Alternatively, we can rank the quantities produced at two different nodes by constructing the sequence $\mathbf{f}$.

rank	c_m	$(c_m)_i$
0	$\frac{1}{2}\mathbf{e}$	$\frac{1}{2}$,
1	$\frac{\lambda}{4}\mathbf{Ge}$	$\frac{\lambda}{4}\deg i$,
2	$\frac{\lambda^2}{8}\mathbf{G}^2\mathbf{e}$	$\frac{\lambda^2}{8}\phi_i^2$,
3	$\frac{\lambda^3}{16}\mathbf{G}^3\mathbf{e}$	$\frac{\lambda^3}{16}\phi_i^3$,

where we recall that ϕ_i^k denotes the total number of paths of length k originating from i in network $\mathbf{G}$. Hence, at the first order, quantities are higher (respectively lower) at nodes with higher degrees when externalities are positive (respectively negative). Finally, as there are no global effects, the effect of the addition of a link between i and j on the quantity produced at node k can easily be ascertained, using the discussion following Proposition 3. If externalities are positive, the quantity produced at k always increases ; if externalities are negative, the quantity produced increases if and only if the minimal distance between k and either i or j is even.

3.2 Bertrand competition

In order to study price competition, we first invert the system of inverse demand equations (5) to construct a system of demand equations:

$$\mathbf{q} = (\mathbf{I} - \lambda\mathbf{G})^{-1}(\mathbf{e} - \mathbf{p}), \tag{8}$$

so that

$$q_i = \sum_j a_{ij} - \sum_j a_{ij} p_j$$

for the matrix $\mathbf{A} = [a_{ij}] = (\mathbf{I} - \lambda\mathbf{G})^{-1}$. Taking first order conditions, we obtain:

$$2a_{ii}p_i + \sum_{j \neq i} a_{ij} p_j = \sum_j a_{ij}.$$

Alternatively, we can characterize the equilibrium prices as solutions to the system of linear equations:

$$(\Delta((\mathbf{I} - \lambda\mathbf{G})^{-1}) + (\mathbf{I} - \lambda\mathbf{G})^{-1})\mathbf{p} = ((\mathbf{I} - \lambda\mathbf{G})^{-1})\mathbf{e}$$

or

$$\begin{aligned}
(\Delta((\mathbf{I}-\lambda\mathbf{G})^{-1}) + (\mathbf{I}-\lambda\mathbf{G})^{-1})\mathbf{p} - c\mathbf{e}) &= (I-\lambda\mathbf{G})^{-1}\mathbf{e}, \\
(I-\lambda\mathbf{G})(\Delta((\mathbf{I}-\lambda\mathbf{G})^{-1}) + (\mathbf{I}-\lambda\mathbf{G})^{-1})\mathbf{p} &= \mathbf{e}, \\
(\mathbf{I} + (\mathbf{I}-\lambda\mathbf{G})(\Delta((\mathbf{I}-\lambda\mathbf{G})^{-1})))\mathbf{p} &= \mathbf{e},
\end{aligned}$$

where the first equality is obtained by premultiplying both sides of the equation by $(I-\lambda\mathbf{G})$, and the second equality by noting that $(I-\lambda\mathbf{G})(\Delta((\mathbf{I}-\lambda\mathbf{G})^{-1}) + (\mathbf{I}-\lambda\mathbf{G})^{-1}) = (\mathbf{I} + (\mathbf{I}-\lambda\mathbf{G})(\Delta((\mathbf{I}-\lambda\mathbf{G})^{-1})))$. Having completed this characterization, we now have a system of linear equations characterizing equilibrium prices p^* where

$$\mathbf{I}-\mathbf{A} = (\mathbf{I} + (\mathbf{I}-\lambda\mathbf{G})(\Delta((\mathbf{I}-\lambda\mathbf{G})^{-1})))$$

and $a = \mathbf{e}$.

In order to analyze the matrix $\mathbf{I}-\mathbf{A}$, we apply the asymptotic methods developed in the previous Section, and study the prices which arise when external effects vanish. Consider a sequence $\{\beta_t\}$ converging to 0. As in Proposition 1, we use the power series expansion to write:

$$(\mathbf{I}-\beta_t\lambda\mathbf{G})^{-1} = \sum_{k=0}^{\infty} \beta_t^k \lambda^k \mathbf{G}^k.$$

and the diagonal terms of this matrix can be written:

$$\xi_{ii} = \sum_{k=0}^{\infty} \beta_t^k \lambda^k \nu_i^k,$$

where we recall that ν_i^k denotes the number of paths of length k in $\mathbf{G}$ from i to i. We can thus write

$$\begin{aligned}
\mathbf{I}-\mathbf{A} &= \mathbf{I} + (\mathbf{I}-\beta_t\lambda\mathbf{G}) \sum_{k=0}^{infty} \beta_t^k \lambda^k \Delta(\nu_i^k), \\
&= 2\mathbf{I} - \beta_t\lambda\mathbf{G} + \sum_{k=2}^{\infty} \beta_t^k \lambda^k (\Delta(\nu_i^k) - \Delta(\nu_i^{k-1})\mathbf{G}).
\end{aligned}$$

We thus construct the sequence

$$
\begin{aligned}
B_1 &= \mathbf{G}, \\
B_2 &= -\Delta(\mathbf{d}), \\
&\dots \\
B_k &= -(\Delta(\nu_i^k) - \Delta(\nu_i^{k-1})\mathbf{G}),
\end{aligned}
$$

Applying Proposition 2, we compute the first terms of the sequence $\mathbf{f}$ in order to compare equilibrium markups at different nodes.

rank	c_m	$(c_m)_i$
0	$\frac{1}{2}\mathbf{e}$	$\frac{1}{2}$,
1	$\frac{\lambda}{4}\mathbf{d}$	$\frac{\lambda}{4}\deg i$,
2	$\frac{\lambda^2}{8}(\mathbf{G}^2\mathbf{e} - \mathbf{d})$	$\frac{\lambda^2}{8}\sum_j g_{ij}(\deg j - 1)$,

This computation shows that, as in the case of Cournot competition, at the first order, the relevant characteristic of node i is its degree. Prices will be higher for consumers with larger degrees if externalities are positive, smaller if externalities are negative. At first glance, this result may seem counterintuitive. One may think that prices and quantities are negatively correlated, so that nodes where Cournot competitors produce higher quantities should also be nodes where Bertrand competitors charge lower prices. In our model, this is not the case for the following reason. The firm serving node i does not realize that the price p_i affects demand on neighboring nodes. Taking the prices and demand on neighboring nodes as given, the firm serving node i only observes that consumption externalities are higher when node i has a higher degree, and hence charges a higher price. Cournot competitors also ignore the effect that their choices have on neighboring nodes, and serve larger quantities at nodes with higher degree, where consumption externalities are higher.

If two nodes have the same degree, the next component to consider in the lexicographic ordering is the sum of the degree of the agent's neighbors: the higher this measure is, the higher the price both for positive and negative consumption externalities.

As there are no global effects in the equation ($\gamma = 0$), Proposition 3 allows us to study the effect of a change in the network on the prices charged to all consumers. The creation of a link between agents i and j will start affecting agent k in the power matrix G^ζ, where ζ denotes the minimal distance between k and either i or j. Hence, the first vector for which $(\mathbf{c}_m)_k \neq (\mathbf{c}'_m)_k$ is the vector $\mathbf{c}_\zeta$. Using Proposition 3, we conclude that the formation of a new link ij increases all prices when externalities are positive, and results in an increase in price p_k if and only if the minimal distance between k and either i or j is even when externalities are negative.

3.3 Collusion

We now turn our attention to the situation where prices are chosen in a collusive fashion. The optimal prices chosen by a monopolist satisfy:

$$2\sum_j a_{ij} p_j = \sum_j a_{ij},$$

where $\mathbf{A} = [a_{ij}] = (\mathbf{I} - \lambda \mathbf{G})^{-1}$. It is easy to see that this system of equations has a unique symmetric solution, $p_i = p_j = \frac{1}{2}$. Hence, the multiproduct monopolist, internalizing the local network externalities, *charges the same monopoly price at each node.*

While prices are identical across nodes, the quantities sold will differ. Quantities are given by the solution to the system of equations:

$$q_i - \lambda \sum_j g_{ij} q_j = \frac{1}{2}.$$

This is a simple system of linear equations with $\mathbf{I} - \mathbf{A} = -\lambda \mathbf{G}$, (so $\delta = 1$ and $\mathbf{B}_1 = \mathbf{G}$), and we conclude that quantities are proportional to the Bonacich centrality measure of each node with scalar λ.

4 Other Applications

4.1 Average price externalities

We now consider a model where agents compare the price they receive with that of their neighbors. Suppose that agents have utilities defined over the average price charged to their neighbors:

$$U_i = \theta_i - p_i + \lambda \frac{1}{\deg i} \sum_j g_{ij} p_j.$$

where θ_i is a taste parameter uniformly distributed on $[0, 1]$. If $\lambda > 0$, the model captures the pleasure a consumer experiences by buying the good at a price below the average price charged to his neighbors. This is a social component of utility that is likely to be observed when consumers buy durable goods, real estate, holiday packages, etc..

A consumer located at node i buys the good if and only if

$$\theta_i \geq p_i - \lambda \frac{1}{\deg i} \sum_j g_{ij} p_j,$$

Hence the demand at node i is given by:

$$q_i = 1 - p_i + \lambda \frac{1}{\deg i} \sum_j g_{ij} p_j.$$

This results in equilibrium prices which satisfy:

$$2p_i - \lambda \frac{1}{\deg i} \sum_j g_{ij} p_j = 1, \tag{9}$$

or in matrix terms:

$$(2\mathbf{I} - \lambda \Delta(\mathbf{id})\mathbf{G})\mathbf{p} = \mathbf{e}. \tag{10}$$

Notice that the matrix $\Delta(\mathbf{id})\mathbf{G}$ is a stochastic matrix. Hence, e is an eigenvector of the matrix with associated eigenvalue 1, and

$$(\mathbf{I} - \frac{\lambda}{2} \Delta(\frac{1}{\deg i})\mathbf{G}) \frac{1}{2-\lambda} e = \frac{}{2} e,$$

Hence, *prices are equal across locations* and $p^* = \frac{1}{2-\lambda} e$. This striking result shows that, when each node is served by a different firm, the "relative comparison" externality does not result in differences in the prices charged by the firms.

The results are very different if one considers a multiproduct monopolist. In that case, the first order condition yields:

$$2p_i - \lambda(\frac{1}{\deg i} \sum_j g_{ij} p_j - \sum_j g_{ij} \frac{1}{\deg j} p_j = 1, \tag{11}$$

or in matrix terms

$$(2\mathbf{I} - \lambda(\Delta(\mathbf{id})\mathbf{G} + \mathbf{G}\Delta(\mathbf{id}))\mathbf{p} = \mathbf{e}. \tag{12}$$

We thus have $\delta = 2$, $\mathbf{B}_1 = (\Delta(\mathbf{id})\mathbf{G} + \mathbf{G}\Delta(\mathbf{id}))$ and $\mathbf{a} = \mathbf{e}$, enabling us to compute the first terms of the sequence $\mathbf{f}$:

rank	c_m	$(c_m)_i$
0	$\frac{1}{2}\mathbf{e}$	$\frac{1}{2}$,
1	$\frac{\lambda}{4}(\mathbf{e}+\mathbf{Gid})$	$\frac{\lambda}{4}(1+\sum_j g_{ij}\frac{1}{\deg j})$,
2	$\frac{\lambda^2}{8}(\mathbf{e}+\mathbf{id}+\Delta(\mathbf{id})\mathbf{G}^2\mathbf{id}+\mathbf{G}\Delta(\mathbf{id})\mathbf{Gid})$	$\frac{\lambda^2}{8}(1+\frac{1}{\deg i}+\frac{1}{\deg i}(\sum_{j,k} g_{ij}g_{jk}\frac{1}{\deg k})$ $+\sum_j g_{ij}\frac{1}{\deg j}\sum_k g_{jk}\frac{1}{\deg k})$.

We observe that a multiproduct monopolist exploits the relative comparison externality and charge different prices at different nodes. Interestingly, the ranking between p_i and p_j in the first order *does not depend on the degrees of i and j* but on the sum of the inverse of the degrees of their neighbors. In words, price p_i will exceed price p_j if consumer i is surrounded by a larger number of neighbors with smaller degrees than consumer j. According to this measure, it is clear that the highest price will be charged to the hub in a star (which has a large number of neighbors with the smallest degree) and the lowest price to a peripheral agent in a star (who has the smallest number of neighbors with the largest degree.) Finally, note that, because $\mathbf{A}$ is a nonnegative matrix, an increase in the interaction network results in an overall increase in price, and that the prices charged by the monopolist always exceed the prices charged by competing firms.

4.2 Crime networks

Calvo-Armengol and Zenou (2004) and Ballester, Calvo-Armengol and Zenou (2004) study a model where criminals are embedded in a social network and choose strategically the amount of criminal activities, x_i. They suppose that the benefit to a criminal can be decomposed as :

$$U_i = y_i - p_i,$$

where y_i is the reward to the criminal activity and p_i the punishment incurred. Furthermore, they suppose that the rewards to criminal activities depend on *total efforts* made by all criminals whereas the punishment is a function of the *local efforts* made by neighbors,

$$\begin{aligned} y_i &= x_i(1-\sum_j x_j), \\ p_i &= \lambda x_i(1-\sum_j g_{ij}x_j), \end{aligned}$$

Under these assumptions, the utility of a player is a quadratic function:

$$U_i = x_i(1-\lambda) - x_i^2 - \sum_j x_i x_j + \lambda \sum_j g_{ij} x_j. \tag{13}$$

Assuming that λ is small enough, the unique interior equilibrium will be characterized by the first order conditions:

$$2x_i - \sum_j x_j + \lambda \sum_j g_{ij} x_j = 1 - \lambda, \tag{14}$$

or

$$(2\mathbf{I} - \mathbf{U} + \lambda \mathbf{G})\mathbf{x} = (1-\lambda)\mathbf{e}. \tag{15}$$

We may thus compute the vectors c_m using $\delta = 2$, $\mathbf{B}_1 = \mathbf{G}$, $a_0 = \mathbf{e}$, and $a_1 = -\mathbf{e}$, to find:

rank	c_m	$(c_m)_i$
0	$\frac{1}{2}\mathbf{e}$	$\frac{1}{2}$,
1	$\frac{\lambda}{2}\mathbf{e} + \frac{\lambda}{4}\mathbf{d}$	$\frac{\lambda}{2} + \frac{\lambda}{4}\deg i$,
2	$\frac{\lambda^2}{4}\mathbf{d} + \frac{\lambda^2}{8}\mathbf{G}^2\mathbf{e}$	$\frac{\lambda^2}{4}\deg i + \frac{\lambda^2}{8}\phi_i^2$.

Hence, at the first order, the amount of critical activities is related to a player's degree: criminals with more neighbors will exert more efforts. At the second order, the amount of critical activities is related to ϕ_i^2, the number of paths of length 2 originating from player i. In fact,the effort exerted by every criminal is proportional to his Bonacich centrality measure – as observed by Ballester, Calvo-Armengol and Zenou (2004). Our calculations just enable us to check that fact.

4.3 R & D Networks

Goyal and Moraga Gonzalez (2001) study a model of strategic alliances in oligopolies, where firms choose endogenously their amount of R & D effort. A firm can gain access to a fraction λ of the research output of another firm by forming a strategic alliance. Hence, the constant marginal cost of production of firm i is given by:

$$c_i = \overline{c} - x_i - \lambda \sum_j g_{ij} x_j. \tag{16}$$

The cost of research efforts is quadratic given by $c(x_i) = \eta x_i^2$. Suppose that the firms are Cournot competitors selling a homogeneous product. Then standard calculations show that the equilibrium quantity is given by:

$$q_i = \frac{1}{n+1} - c_i + \frac{\sum_j c_j}{n+1} \tag{17}$$

so that equilibrium profit is given by:

$$\pi_i = (\frac{1}{n+1} - c_i + \frac{\sum_j c_j}{n+1})^2 - \eta x_i^2, \tag{18}$$

$$= (\frac{1-\bar{c}}{n+1} + x_i + \lambda \sum_j g_{ij} x_j - \frac{\sum_j x_j}{n+1} - \lambda \sum_j \sum_k \frac{g_{jk} x_k}{n+1})^2 - \eta x_i^2 \tag{19}$$

The first order conditions for an interior equilibrium yield:

$$\begin{aligned} x_i(\eta(n+1)^2 - n(n+1)) + n \sum_j x_j + \lambda_n \sum_j x_j \deg j \\ + \lambda((n+1)\deg i x_i - n(n+1)\sum_j g_{ij} x_j - \deg i \sum_j x_j) \\ + \lambda^2(\deg i \sum_j g_{ij} x_j + \deg i \sum_j x_j \deg j) &= n(1-\bar{c}) - \lambda(1-\bar{c})\deg i. \end{aligned}$$

From this complex system, eliminating the global effects, we compute $\delta = \eta(n+1)^2 - n(n+1), \mathbf{B}_1 = (n+1)\Delta(\mathbf{d}) - \Delta(\mathbf{d})\mathbf{U} - n(n+1)\mathbf{G}, \mathbf{B}_2 = (n+1)\Delta(\mathbf{d})\mathbf{G} - \Delta(\mathbf{d})\mathbf{U}\Delta(\mathbf{d}), \mathbf{a}_0 = n(1-\bar{c})\mathbf{e}$ and $\mathbf{a}_1 = -(1-\bar{c})\mathbf{d}$.
Focussing on the first two terms of the sequence $\mathbf{f}$, we compute:

rank	c_m	$(c_m)_i$
0	$\frac{n(1-\bar{c})}{\eta(n+1)^2-n(n+1)}\mathbf{e}$	$\frac{n(1-\bar{c})}{\eta(n+1)^2-n(n+1)}$,
1	$-\frac{\lambda(1-\bar{c})}{2(\eta(n+1)^2-n(n+1))}\mathbf{d} - \frac{\lambda n(n^2-n-1)(1-\bar{c})}{(\eta(n+1)^2-n(n+1))^2}\mathbf{d}$,	$-(\frac{\lambda(1-\bar{c})}{2(\eta(n+1)^2-n(n+1))} + \frac{\lambda n(n^2-n-1)(1-\bar{c})}{(\eta(n+1)^2-n(n+1))^2})\deg i$,

Our calculations thus show that, at the first order, the effort chosen by a firm is decreasing in its degree. Firms with more strategic alliances exert less R & D efforts. Note that this asymptotic result was obtained for any network structure. By contrast, Goyal and Moraga Gonzales (2001) restrict their study to symmetric networks (where they show that R &D efforts are nonmonotonic in degree and reach a maximum for interior values of the degree), and to networks among three firms.

5 Sensitivity analysis

As our results are based on approximations of the solutions to systems of linear equations, it is important to understand how parameters of the model affect the accuracy of the approximations. To this end, we report here on simulations which were conducted for the Bertrand competition model of optimal pricing with consumption externalities among five nodes.

We consider five different network structures corresponding to all the connected networks (up to a permutation of the agents), namely: the circle (network 1), the star where agent 2 is the hub (network 2), the complete network (network 3), the network where three agents (namely, agents 1,2 and 3) form a triangle and the last agent is linked to agent 1 only (network 4), the last structure (network 5) is characterized by the following matrix:

$$G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

For each structure two approaches are considered. First the exact equilibrium prices are evaluated, then we compute the values of components c_1 and c_2 used in the previous sections.

Network 1

We computed the exact values of optimal prices for different intensities of the external effects by using the expressions derived in previous sub sections. The results are as follows:

	Agent 1	Agent 2	Agent 3	Agent 4
$(c_1)_i$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$(c_2)_i$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
p_{ex}, $\beta = 0.01$	0.505	0.505	0.505	0.505
p_{ex}, $\beta = 0.05$	0.525	0.525	0.525	0.525
p_{ex}, $\beta = 0.10$	0.5506	0.5506	0.5506	0.5506
p_{ex}, $\beta = 0.25$	0.6316	0.6316	0.6316	0.6316

As expected there is only one price charged (for a given intensity of external effects) since the agents' characteristics are the same.

Network 2

We computed the exact values of optimal prices for different intensities of the external effects by using the expressions derived in previous sub sections. The results are as follows:

	Agent 1	Agent 2	Agent 3	Agent 4
$(c_1)_i$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$(c_2)_i$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$
p_{ex}, $\beta = 0.01$	0.5025	0.5075	0.5025	0.5025
p_{ex}, $\beta = 0.05$	0.5129	0.5365	0.5129	0.5129
p_{ex}, $\beta = 0.10$	0.5267	0.5709	0.5267	0.5267
p_{ex}, $\beta = 0.25$	0.579	0.6579	0.579	0.579

Agent 2 is charged the highest price as his degree is larger than those of the other agents. Because the other agents' characteristics are the same, they are charged the same price.

Network 3

We computed the exact values of optimal prices for different intensities of the external effects by using the expressions derived in previous sub sections. The results are as follows:

	Agent 1	Agent 2	Agent 3	Agent 4
$(c_1)_i$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
$(c_2)_i$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
p_{ex}, $\beta = 0.01$	0.5075	0.5075	0.5075	0.5075
p_{ex}, $\beta = 0.05$	0.5385	0.5385	0.5385	0.5385
p_{ex}, $\beta = 0.10$	0.579	0.579	0.579	0.579
p_{ex}, $\beta = 0.25$	0.7142	0.7142	0.7142	0.7142

Again, the situation is symmetric, and all agents are charged the same price for a given intensity of external effects. However, one can notice that these agents are charged higher prices compared to the situation described by the first network. The difference here is that the agents' degrees are strictly higher. As expected, this results in higher values of equilibrium prices.

Network 4

We computed the exact values of optimal prices for different intensities of the external effects by using the expressions derived in previous sub sections. The results are as follows:

	Agent 1	Agent 2	Agent 3	Agent 4
$(c_1)_i$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
$(c_2)_i$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{4}$
p_{ex}, $\beta = 0.01$	0.5075	0.505	0.505	0.5025
p_{ex}, $\beta = 0.05$	0.5371	0.5253	0.5253	0.5129
p_{ex}, $\beta = 0.10$	0.5735	0.5515	0.5515	0.527
p_{ex}, $\beta = 0.25$	0.6723	0.6384	0.6384	0.5853

The situation can be decomposed into three different sub sets of agents who are diversely affected by local effects. First, agent 1 is characterised by the highest degree and is charged the highest price. Second, agents 2 and 3 have the same characteristics and are charged the same price. Finally, agent 4 has the lowest degree and is charged the lowest price.

Network 5

We computed the exact values of optimal prices for different intensities of the external effects by using the expressions derived in previous sub sections. The results are as follows:

	Agent 1	Agent 2	Agent 3	Agent 4
$(c_1)_i$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
$(c_2)_i$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
p_{ex}, $\beta = 0.01$	0.505	0.5075	0.505	0.5075
p_{ex}, $\beta = 0.05$	0.5257	0.5378	0.5257	0.5378
p_{ex}, $\beta = 0.10$	0.5533	0.5762	0.5533	0.5762
p_{ex}, $\beta = 0.25$	0.6586	0.6887	0.6586	0.6887

In the present case there are two sub sets of agents, the first one where agents 1 and 3 have two neighbors who are the members of the second subset, namely agents 2 and 4, who are connected to all other agents. As such, they are charged a higher price than agents 2 and 3.

6 Conclusions

In this paper, we study optimal pricing and strategic decisions in networks with quadratic objective functions, following Ballester, Calvo-Armengol and Zenou (*Econometrica*, 2006). We focus on two questions: How do optimal prices and strategies reflect the position of agents in the network? What is the effect of a change in the network structure on optimal pricing and strategic decisions? Using an asymptotic approach, we show that, when local effects become small, the ranking of optimal prices and strategies can be reduced to the ranking of simple characteristics of the agent's position in the network. Furthermore, in the absence of global effects, the addition of a link between two agents i and j in the network affects the equilibrium price or strategy of agent k according to the minimum distance between k and i or j in the network.

The contribution of this paper is primarily methodological: we propose an asymptotic approach to study comparative statics effects which would otherwise be impossible to sign. In order to test the usefulness of this approach, we consider different economic applications where traditional methods fail to produce results. In a model of optimal pricing with local network externalities, we show that optimal prices are increasing in a node's degree. In a model of strategic alliances, we establish that R & D efforts are decreasing in a firm's number of alliances. In a model where consumers compare the prices they receive with the average price in the neighborhood, we observe that prices depend on the sum of the inverses of the degrees of a node's neighbor.

Of course, we are aware of the limitations of our analysis. As we consider approximations for small local effects, our method cannot be used to analyze model with large network effects. Our simulations show that, for a small number of agents, the ranking of optimal decisions obtained for small effects remains true for higher values of local effects. However, a more systematic sensitivity analysis is needed to measure how our results carry over to a broader range of values of the magnitude of local effects. We have also focussed attention on the effect of changes in the connectivity of the network on equilibrium prices and decisions. Following Galeotti and Goyal (2007), we may also look at "second order stochastic dominance" effects, where the number of links in the network is kept fixes, but the variance of the degree distribution is reduces.

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8 Proofs

Proof of Proposition 1: Recall that $||\mathbf{A}||$ is *not* a matrix norm, but $n||\mathbf{A}||$ satisfies the submultiplicity condition, and is indeed a matrix norm (Horn and Johnson (1986), Example 5, p. 322). Hence, if $n||\mathbf{A}|| < 1$, the power series $\sum_k \mathbf{A}^k$ is convergent in one matrix norm, so that $\mathbf{I} - \mathbf{A}$ is invertible and

$$(\mathbf{I} - \mathbf{A})^{-1} = \sum_{k=0}^{\infty} \mathbf{A}^k.$$

We thus have:

$$\mathbf{x}^* = \sum_{k=0}^{K} \mathbf{A}^k \mathbf{a}.$$

Now,

$$\begin{aligned} ||\mathbf{x}^* - \sum_{k=0}^{K} \mathbf{A}^k \mathbf{a}|| &= ||\sum_{k=K+1}^{\infty} \mathbf{A}^k \mathbf{a}|| \\ &\leq \sum_{k=K+1}^{\infty} ||\mathbf{A}^k \mathbf{a}|| \\ &\leq \sum_{k=K+1}^{\infty} n||\mathbf{A}^k||||\mathbf{a}|| \\ &\leq \sum_{k=K+1}^{\infty} n^k ||\mathbf{A}||^k ||\mathbf{a}||, \\ &\leq \frac{n^{K+1}||\mathbf{A}||^{K+1}||\mathbf{a}||}{1 - n||\mathbf{A}||}. \end{aligned}$$

The first inequality derives from the triangle inequality of the vector norm, the second from the fact that the matrix norm $n||\mathbf{A}||$ is compatible with the vector norm $||\mathbf{A}||$ (Horn and Johnson (1986), Theorem 5.7.13 p. 324) and the third from the fact that the matrix norm $n||\mathbf{A}||$ is submultiplicative.

Proof of Proposition 2: As in BCAZ, we first observe that, when comparing the strategic decisions at nodes i and j, we can safely ignore the global effect. In fact, recall that, by the decomposition of the matrix $\mathbf{I} - \mathbf{A}$,

$$\delta x_i + \gamma \sum x_j - \lambda \sum b_i j x_j = \alpha_i. \tag{20}$$

Consider the solution to an alternative system of linear equations, where the global effects are ignored,

$$\delta x_i - \lambda \sum b_i j x_j = \alpha_i, \tag{21}$$

Clearly, the difference between two components x_i^* and x_j^* is equal in the two systems, so we ignore the uniform effects and consider a system of linear equations with $\gamma = 0$. Furthermore, we can now renormalize the system of linear equations,

$$x_i - \frac{\lambda}{\delta} \sum b_i j x_j = \frac{\alpha_i}{\delta_i}, \tag{22}$$

a new system of equations with a matrix $\mathbf{A}' = \frac{\lambda}{\delta}\mathbf{B}$ and a vector $\mathbf{a}' = \frac{\mathbf{a}}{\delta}$. Observe that the sequence c_m has be defined exactly so that

$$\sum_{k=0}^{\infty} \mathbf{A}'^k \mathbf{a} = \sum_{m=1}^{\infty} \beta^m c_m.$$

Because the sequence of matrices $\mathbf{B}_t$ converges to the zero matrix, there exists $T > 0$ such that $||\mathbf{A}'_t|| \leq \frac{1}{n}$ for all $t \geq T$, and by Proposition 1, the system of linear equations $(\mathbf{A}'_t, \mathbf{a}'_t)$ possesses a unique solution. Furthermore, for $t \geq T$, the series $\sum_{k=0}^{\infty} \mathbf{A}'^k_t \mathbf{a}'_t$ is convergent, so that $\sum_{m=0}^{\infty} \beta_t^m ||c_m||$ is a convergent series.
Next, using Proposition 1, recall that

$$||x_t^* - \sum_{m=0}^{M} \beta_t^m \mathbf{c}_m|| \leq || \sum_{m=M+1}^{\infty} \beta_t^m \mathbf{c}_m||.$$

By definition of the l_∞ vector norm, this implies that for all $i = 1, 2, ...n$,

$$|(x_t^*)_i - \sum_{m=0}^{M} \beta_t^m (\mathbf{c}_m)_i| \leq || \sum_{m=M+1}^{\infty} \beta_t^m \mathbf{c}_m||. \tag{23}$$

Now consider a pair (i, j) and let M be the first element of the sequences $\mathbf{f}_i$ and $\mathbf{f}_j$ such that $(\mathbf{c}_M)_i \neq (\mathbf{c}_M)_j$. Applying equation (23) to i and j, we obtain,

$$|\frac{(x_t^*)_i - (x_t^*)_j}{\beta_t^M} - ((\mathbf{c}_M)_i - (\mathbf{c}_M)_j)| \leq 2\beta_t || \sum_{m=M+1}^{\infty} \beta_t^{m-M-1} \mathbf{c}_m ||. \tag{24}$$

Now, because the series $\sum_{m=0}^{\infty} \beta_T^m ||\mathbf{c}_m||$ is convergent, there exists a positive F such that:

$$\sum_{m=0}^{\infty} \beta_T^m ||\mathbf{c}_m|| \leq F,$$

so that

$$\sum_{m=M+1}^{\infty} \beta_T^{m-M-1} ||\mathbf{c}_m|| \leq \frac{F}{\beta_T^{M+1}}. \tag{25}$$

Now,

$$\begin{aligned} || \sum_{m=M+1}^{\infty} \beta_t^{m-M-1} \mathbf{c}_m || &\leq \sum_{m=M+1}^{\infty} \beta_t^{m-M-1} ||\mathbf{c}_m||, \\ &\leq \sum_{m=M+1}^{\infty} \beta_T^{m-M-1} ||\mathbf{c}_m||, \\ &\leq \frac{F}{\beta_T^{M+1}}, \end{aligned}$$

where the first inequality stems from the properties of the vector norm, the second inequality from the fact that β_t converges *monotonically* to zero, and the last inequality from equation (25). Now, this implies that, for any $\epsilon > 0$ there exists $T' > 0$ such that, for all $t \geq \max\{T, T'\}$,

$$2\beta_t || \sum_{m=M+1}^{\infty} \beta_t^{m-M-1} \mathbf{c}_m || \leq \epsilon \frac{F}{\beta_T^{M+1}},$$

Hence, by inequality (24) the difference between $\frac{(x_t^*)_i - (x_t^*)_j}{\beta_t^M}$ and $((\mathbf{c}_M)_i - (\mathbf{c}_M)_j)$ can be made arbitrarily small, concluding the proof of the Proposition.

Proof of Proposition 3: Omitted as it is similar to the proof of Proposition 2.