

Stability and Social Recognition of Authority: Collective Production in a Pre-Market Setting

Emiliya A. Lazarova* Robert P. Gilles† Pieter H.M. Ruys‡

May 14, 2007

1 On Social Complexity and Productive Complexity

Complex economic organizations, such as multi-hierarchy firms are taken to be a common economic entity. Coase (1937) was the first to question the origin of the firm and by raising this question he started a substantial volume of literature that discusses the organizational form of the firm as an alternative to the market organizations. However, complex productive organizations have emerged even before the establishment of markets as institutions. While such pre-market organizations have been ignored in the above works, they will be in the center of attention here. The current work develops along two dimensions. On one hand, we develop a general framework that allows us to study the emergence of complex productive entities, and on the other hand, we apply this framework to a specific collective production process, via which, according to anthropologists, complexity emerged in primitive hunter-gatherer societies.

We develop a theoretical framework in which economic value is generated in the interaction between people. An individual has a restricted domain of activities in which she can engage. The restriction may be coming from exogenously given social or production-specific rules. Given their potential domain, individuals are seeking to activate those relations, that allow them to participate in a productive process. The productive processes studied here may involve more than a pair of individuals, that is why we refer to them as complex processes. What is essential is that these individuals who engage in a productive process are organized in a specific way that follows from our definition of complexity. Complexity in an organization is translated in graph-theoretic terms as a directed network of star structures that we call a *star pattern*. The direction of a productive link signifies that one of the individuals has authority over the other. The utility of an individual engaged in production depends on her productive ability and on the abilities of the other players with whom she is connected as well as her relative position in the star, *i.e.*, whether she is located in the center or on one of the leaves. In the general framework, however, we do not explicitly model utility as a function of productivity abilities but use a *hedonic team utility profile* as an indirect utility measure. This

*School of Management and Economics, Queen's University Belfast, 25 University Square, BT7 1NN. Belfast, Northern Ireland. Email: e.lazarova@uqub.ac.uk

†Department of Economics, Virginia Polytechnic Institute & State University, Blacksburg, VA 24061, USA. Email: rgilles@vt.edu

‡Center for Economic Research and Department of Econometrics, Tilburg University, Warandelaan 2, P.O. Box 90153, 5000 LE Tilburg, The Netherlands. Email: ruys@uvt.nl

makes our main theoretical results applicable to a wide range of frameworks. The potential activities and the preferences of players over the production teams in which they can engage define what we call a *team economy*.

We study two types of stability: stability within a specific team economy, and *generic stability* that renders stability to the structure of potential activities. Generic stability implicitly requires that a structure of potential activities is such that no matter how the productive abilities are distributed among its individuals, there is a stable star (production) pattern. The lack of a stable star pattern is interpreted as chaotic behavior and unpredictability of the production outcome given a structure of potential activities and a distribution of the productive abilities. In terms of the individual possibilities for deviation, stability refers to the lack of pairs of players who prefer to deviate from the established pattern. This notion of stability is largely based on the notion of pairwise stability first introduced by Jackson and Wolinsky (1996). Here, however, the notion of stability is adapted to allow for greater possibilities for deviation of these players who are located in the center of a star component. Such players may add links, without severing existing links, unlike the rest of the players who are located at the leaves of a star, because in doing so they still preserve the star structure of the activity pattern.

Our main theoretical results refer to two types of stability: economy-specific and generic. Generic stability is a property of the potential structure that ensures the existence of a stable pattern of a star structure for any preference profile. We provide a sufficient condition on the set of potential activities that ensures the existence of a stable star pattern in a given economy. Furthermore, we provide necessary and sufficient conditions on the set of potential activities that ensure the existence of stable star patterns for any team utility profile, *i.e.*, that ensure generic stability. These necessary and sufficient conditions require that the set of potential links does not have a cycle with a number of players different from a multiple of three, where the multiple is higher than one.¹ The implications of these conditions for the structure of potential activities is, that it may be a tree or it may contain special cycles which connect some players whose number is a multiple of three. Though related to the main theorem on existence of stable coalition structures derived by Pápai (2004)²

The relation between the establishment of a hierarchical social structure and productive complexity via labor specialization has been documented in some anthropological works. This motivated us to develop an application of the general framework closely related to the discussion of the emergence of complexity in primitive societies carried out in some anthropological literature.

In some of anthropological works hierarchical organization is seen as an element of complexity, *e.g.*, in Arnold (1993) we read

I define chiefly complexity to include three recognizable organizational characteristics: hereditary inequality, hierarchical organization (including some political

¹The condition on the number of players connected in a cycle is a technical restriction that is founded on the limited types of complex patterns, which we investigate. We will come back to this point in the discussion in Section 7.

²our theoretical results are fundamentally different, however, and can only be derived in a graph-theoretic framework. The graph-theoretic framework allows us to distinguish between coalitions of, for example, three players in which there are links between all three of them or in which there are links between only two of them and the player in the middle acts as a star central player.

authority on a multicomcommunity scale), and the elite ability to exercise partial control over domestic labor.

The hereditary property of the hierarchical organization implies that the hierarchical organization is ex-ante fixed on a set potential of activities. That is, when individuals decide on the type of activities to do, they are limited by their relative position in the socially recognized hierarchy. Note that here the hierarchical organization refers to a social hierarchy, not necessarily induced by the productive capabilities of individuals. The third element in Arnold's definition of complexity is what we define as a complex value generating relation, *i.e.*, the relation in which an individual has decision power on the production units intrinsic to another individual.

The process via which complex relations have emerged has been attributed by some anthropologists to the socially recognized specialization of labor. For an extensive discussion of these theories we again refer to Arnold (1993):

I suggest that the mechanism by which change occurs from egalitarian to non-egalitarian relations involves the changing organization of human labor, specifically the institutionalized separation of some labor and products from sole family or kin-group control into the hands of higher authority.

In a comparative case study of two hunter-gatherer societies in Papua New Guinea, Bedamuni and Kubo, distinguished by the levels of social and productive complexity (Bedamuni exhibit higher level of complexity and Kubo exhibit lower level of complexity), Minnegal and Dwyer (1998) summarize their expectations as follows:

An increase in the investment of labor in particular tasks usually necessitates a change in the allocation of time to those tasks and thus in the organization of labor. We expected Bedamuni to be investing more in subsistence tasks than Kubo but, more importantly, to be working in different ways. We expected to see a clearer division of labor among Bedamuni than among Kubo, a sharper differentiation of roles between sexes and age classes and work more often performed by specialised task groups than by households. In short, we expected a rationalization of subsistence tasks among Bedamuni, increased differentiation of roles within production units.

When studying the process of the emergence of complex activities, one also needs to answer the question what triggered this process. Again we base the discussion on anthropological evidence. According to Arnold (1993), most anthropologists agree that the emergence of more complex production activities was "a product of necessity rather than choice". Among the ultimate triggers according to Arnold are the high population density and resource imbalances. This implies that even though individuals had the technology to engage in complex production, they did not do so, until it was necessary for them to do it in order to survive. An explanation that we put forward is that the engagement in complex collective production involves reliance on the actions of other individuals and, hence, trust in the social responsibility that the other engaged individuals will have to fulfill their duties in a situation in which there are no established enforcement institutions.

Based on these anthropological insights, we analyse value generation processes which take the form of collective production in a subsistence economy of consumers-producers. We

will introduce in more detail collective production in the following section. Here we would like to point out that the implications of our theoretical results on the application conform to the insights present in the anthropological literature. A necessary and sufficient condition for the presence of stable collective production process is that the set of potential activities has to be void of components of *directed cycles* in which the number of individuals connected is different from a multiple of three. Since there is a specific relation between the direction of the network and the team utility function, the condition is based on the definition of a cycle that also takes into account the direction of the network, *i.e.*, there cannot be two individuals who have (indirect) authority over each other. This implies that the set of potential activities must be of a hierarchical structure which indeed implies the presence of a socially recognized authority.

The remainder of this paper is organized as follows. Section 2 discusses the notion of collective production and the presence of authority. Section 3 provides the necessary graph-theoretic toolbox. Section 4 presents the general framework and applies this framework to the application of complex production. Sections 5 and 6 offer our main theoretical results and illustrative examples. Section 7 concludes with a discussion on the benefits and limitations of this framework.

2 Collective Production and Authority

Collective production is a process in which producer-consumers individuals may engage to generate value. Collective production is, thus, a form of an organization that precedes the emergence of firms. A prominent reference for a discussion of firms formed by consumers-producers is the works of Professor Xiaokai Yang. Yang (2003) defines the *firm* to be “a structure of transactions based on the division of labor”. Yang also recognizes that within a firm there is a clear distinction between two roles: the role of an employer and the role of an employee such that the employer has a decision power over the employee’s labor. We refer to the ability of one individual to make a decision of the participation in the productive process by another individual as *authority*. Collective production is modeled as a stable configuration of agents generating value based on the division of labor with a well-defined flow of authority. The main difference between the organization of collective production and a firm as defined by Yang (2003) is that in our setting there are *no markets*. Hence there is no external mechanism or, for that matter, an invisible hand that determines a rate of remuneration in return for one’s labor. Furthermore, the output produced by collective effort is consumed by the participants in the process rather than sold on an external market.

Moreover, in Yang’s analysis of a firm based on producer-consumers there is a uniform and pre-determined relation between the labor specialization and the authority roles of an “employer” or an “employee” that an individual can assume. The allocation of authority roles in our setting is idiosyncratic to each pair of individuals. So, it is possible that one player has authority over another player, while a third player has authority over the first.

The relation between our framework of collective production and Yang’s analysis of a firm, will become clearer in the discussion below. The collective production is based on that of an economy of producer-consumers discussed by Yang (2003). In Chapter 6 of his book Yang offers a stylized framework in which there is a finite set of individuals who consume a consumption good. To produce the consumption good they need both labor and an intermediate

good. To produce the intermediate good they need only labor as an input. There is increasing returns to scale in the production of both goods. Yang's framework concerns the emergence of firms in a market setting and as such it models players who maximize utility and profits taking prices as given. The objective here is to understand the emergence of complexity at a more primitive level prior to the establishment of the market institutions. Players in our framework maximize their free-time subject to sustaining some minimal consumption level called *subsistence level* \bar{y} .

Like Yang, in our application we assume that individuals are homogeneous in their preferences and production possibility set.³ We denote the amount of the consumption good that they produce by $y \in \mathbb{R}_+$. There is a minimum level of the consumption good \bar{y} that individuals must consume, otherwise they perish. To produce the consumption good individuals use as inputs their labor measured as time and some amount $x \in \mathbb{R}_+$ of an intermediate good. The production of the intermediate good involves only labor, *i.e.*, time. Formally the production functions for the intermediate good and the consumption good are given as $x = (l_x)^\alpha$ and $y = x l_y$, where l_x is the amount of time spent in producing x and l_y is the amount of time spent in producing y and $\alpha > 1$ captures the increasing returns to scale in production.⁴ All individuals have preferences for leisure $f \in [0, 1]$ represented by a linear utility function $\phi(f) = f$. They have one unit of time available which they can allocate among having leisure, producing the intermediate good, or producing the consumption good, *i.e.*, $f + l_x + l_y = 1$. Note that the limited resource of time makes the utility function bounded from below and from above, *i.e.*, $\phi(0) \leq \phi(f) \leq \phi(1)$ for $f \in [0, 1]$. We can interpret $\phi(1)$ as the bliss point of an individual and $\phi(0)$ the point of exhaustion.⁵

In situations in which individuals, acting alone, cannot achieve the subsistence consumption level, by necessity, they need to engage in collective production. The most primitive form of collective production is the one in which only pairs of individuals engage. We conjecture that more complex patterns, such as star production patterns, emerge only after the establishment of stable matching patterns. As we have already seen in Gilles, Lazarova and Ruys (2007), a bi-partite set of potential links is a sufficient and necessary condition for the existence of stable matching patterns. Hence, here we take for granted that the economic agents will choose to specialize in one of two roles: a producer of the intermediate good or a producer of the consumption good.

When two individuals, one specialized in the production of the intermediate good and the other specialized in the production of the consumption good decide to engage in collective production to produce two times the subsistence level, they need to resolve the following question: who works how many hours. On Figure 1 is given the reaction functions of two players who need to collectively produce two times the subsistence level, given their production possibility level as described above in a situation in which none of them can produce the subsistence level acting alone. As we see, the two reaction functions completely coincide. Hence, there is a continuum of points along which the two players can achieve the desired

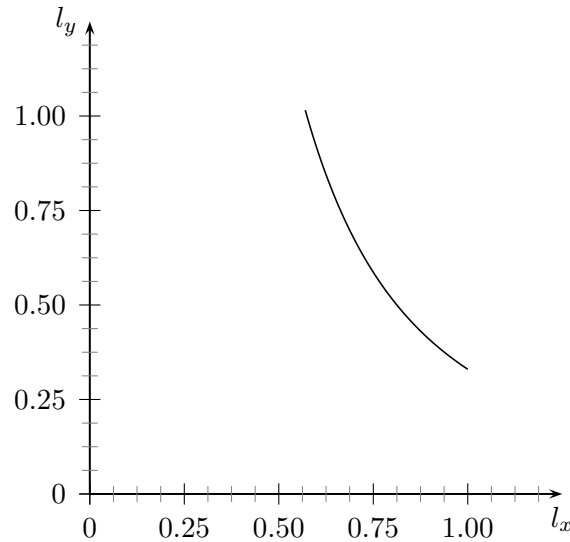
³Since players are homogeneous in their preferences and in their production possibility set we will avoid using individual specific index when no confusion arises.

⁴This production possibility set is analogous to the one discussed by Yang (2003) in the chapter on the emergence of the firm. Yang's formulation is more general, *i.e.*, the production function of consumption good is given by $y = x(l_y)^\beta$. Here we take $\beta = 1$ for ease of exposition.

⁵We have assumed that the utility function of free time is linear for analytical tractability. Alternatively, we could have taken any function defined on the interval $[0,1]$. For instance, we could have taken a cumulative distribution function, which probability density function has thick tails. Such function fits well with the prospect theory described by Kahneman and Thversky (1979).

level of consumption.

Figure 1: Reaction Function of a Pair of Individuals in Collective Production ($\alpha = 2$)



The two end points represent two cases of particular interest: the one in the lower right corner is the case when the producer of the intermediate good devotes all her endowment of time in the production, and the one in the upper left corner is the complementary case when the producer of the consumption good devotes all her time endowment in the production. These cases correspond to the two types of firms analyzed by Yang (2003): the one in which the individual specialized in the production of the consumption good acts as an “employer” and the individual specialized in the production of the intermediate good acts as an “employee”; and the second one in which the individual specialized in the production of the intermediate good acts as an “employer” and the individual specialized in the production of the consumption good acts as an “employee”. In a firm, the individual acting as an “employer” has decision power over the amount of the production input of the “employee”. Hence, an “employer” requires that an “employee” invests all her time endowment into the production. The two possibilities that Yang (2003) discusses are that either all players specialized in the production of the consumption good act as “employers” or players specialized in the production of the intermediate good act as “employers”.

In our analysis we will concentrate on these two cases as well, however, unlike Yang (2003) we will not assume a uniform relation between authority and labor specialization. That is, in our framework whether a person has authority over another person would depend on some exogenous, person-specific characteristics and not on her choice of labor specialization. In other words, it is not necessarily the case that all individuals specialized in the production of, say, the consumption good, have authority over the individuals specialized in the production of the intermediate good. In our framework, it is assumed that one of the individuals has authority over the other in each pair of individuals engaged in collective production, and it is not important what kind of labor specialization the individual has. So, anyone in a pair of players can have authority, what is important is that one of them does.

Clearly, collective production is founded on trust between the participants. One of the

individuals has to submit to the authority of the other. Moreover, she has to trust that she will be provided with the subsistence amount of consumption at the end of the production process. The trust required in establishing collective production explains why collective production is driven by necessity, as pointed out above. If a risk averse individual can achieve the subsistence level acting alone, she will not trust another individual with providing the necessary amount of the consumption good. Under severe circumstances, such as the need to survive, trust is a necessity, and thus, collective production can be established.

Collective production may involve the cooperation between more than two individuals. In our framework, this more complex production process is modeled as a collection of pairs of individuals who have a unique common member. The common member is located in the center of the productive unit and has an additional role of *coordinating* its activities. In this respect, again, our framework is more general than the one developed by Yang (2003) as we do not assume that authority emanates from the individual who has a coordinating role. Instead, we allow for an individual who connects with many other individuals⁶ to be simultaneously under the authority of some and to have authority over others.

3 Technical Preliminaries

Let $N = \{1, \dots, n\}$ be a finite set of economic agents or *players*. Players engage in authoritarian relational activities represented by *directed links* between them, *e.g.*, between two distinct players $i, j \in N$ there may be a directed link (i, j) which indicates that player i has authority over player j and player j does not have authority over player i , *i.e.*, $(i, j) \neq (j, i)$. A directed link between two players refers to a value generating activity between them such that one of the players has authority over the other. In terms of our application, a directed link between two players refers to collective production in which these two players are involved, and the direction runs from the player who has decision power over the labor hours to the other player in the pair. The set of all possible directed links among players in the set N is given as $L_N = \{(i, j) \mid i, j \in N \text{ and } i \neq j\}$. The relational activity represented by the link to oneself we denote by $\{i, i\}$ and we call *autarkic*. An autarkic link is not directed by definition. The set of all autarkic positions is denoted by $D_0 = \{\{i, i\} \mid i \in N\}$.⁷

Definition 3.1 *A set of potential links on the set of individuals N is given as $D \subseteq L_N \cup D_0$ such that*

- $D_0 \subseteq D$;
- for all $i \in N$ there is a distinct player $j \in N$ such that either $(i, j) \in D$ or $(j, i) \in D$;
- $(i, j) \in D$ implies $(j, i) \notin D$.

We define a set of potential links to be an exogenously given subset of the set of all possible relations between the players in N such that no two players have direct authority over each other and that, in addition, contains the autarkic positions for all players. The set of potential links defines the potential activities that can be carried out by players. Since it is a subset

⁶Here only individuals with a different labor specialization can be linked. This is why the only individual in a group who has multiple links has a different specialization from the one of the players with whom she is linked.

⁷Despite the fact that we denote the set of autarkic pairs with the same letter as a set of directed oriented links, we re-emphasize that the set of autarkic pairs is *undirected*.

of the set of all possible links, it is designed to capture physical, institutional, or any other restrictions that may prohibit the occurrence of activities between certain players. A pair (N, D) is called a *directed activity structure*. In terms of our application one can think of the pair (N, D) as a social structure.

Next we introduce some technical definitions from network theory, which will be used later in defining an economy and for deriving the main theoretical results.

Let (N, D) be a directed activity structure. For convenience, when the direction of the link between two distinct players $i, j \in N$ present in D is not important we will use the underlying undirected network Δ corresponding to D , *i.e.*, an undirected link between two players $i, j \in N$ is given by $\{i, j\} \in \Delta$ if $(i, j) \in D$ or $(j, i) \in D$ or $i = j$. Recall that since D is a set of potential links, it cannot be the case that $\{(i, j), (j, i)\} \subseteq D$. We will use the shorthand notation ij for the undirected link $\{i, j\} \in \Delta$.

The location of a player within a network is an important characteristic. Let N be a finite set of players and let $D \subseteq D_N$ be a set of potential links and let Δ be the corresponding undirected network of D . Let $i \in N$ be a player such that there is a distinct player $j \in N$ with $ij \in \Delta$. The *set of connected players* in D is given by $N(D) = \{i \in N \mid \text{there exists } j \in N \text{ with } j \neq i \text{ such that } ij \in \Delta\}$. By definition, since D is a set of potential links $N(D) = N$. We define player i 's *neighborhood* in D as

$$N_i(D) = \{j \in N \mid j \neq i \text{ and } ij \in \Delta\}.$$

We can also express the neighborhood of a player in terms of its link based analogue, *i.e.*,

$$L_i(D) = \{(i, j) \in D \text{ and } (k, i) \in D \mid j \neq i, j \in N_i(D) \text{ and } k \neq i, k \in N_i(D)\}.$$

We define a *path* between any two distinct players $i \in N$ and $j \in N$ in G as a sequence of distinct players $P_{ij}(D) = (i_1, i_2, \dots, i_m)$ with $i_1 = i$, $i_m = j$, $i_k \in N$ and $i_k i_{k+1} \in \Delta$ for all $k \in \{1, \dots, m-1\}$. We call a *cycle* a path of a player from herself to herself which contains at least two other distinct players, *i.e.*, $C(D) = (i_1, i_2, \dots, i_m)$ with $i_1 = i$, $i_m = i$, $m \geq 4$, $i_k \in N$, and $i_k, i_{k+1} \in \Delta$ for all $k \in \{1, \dots, m-1\}$. We also define paths which follow the direction of the network, we call such paths *directed paths*, *i.e.*, a directed path between two distinct players $i \in N$ and $j \in N$ in D is a sequence of distinct players $P_{ij}^d(D) = (i_1, i_2, \dots, i_m)$ with $i_1 = i$, $i_m = j$, $i_k \in N$ and $(i_k, i_{k+1}) \in D$ for all $k \in \{1, \dots, m-1\}$. Similarly, we define a *directed cycle* $C^d(D) = (i_1, i_2, \dots, i_m)$ with $i_1 = i$, $i_m = i$, $m \geq 4$, $i_k \in N$ and $(i_k, i_{k+1}) \in D$ for all $k \in \{1, \dots, m-1\}$. We emphasize that each (directed) cycle has length of at least three, *i.e.*, a cycle consists of at least three distinct links. Note that any directed cycle is a cycle, however, not every cycle is a directed cycle.

Furthermore, there may be players in N between whom there is no path in a set of potential links D ; such players are located in different components of D . A network $d \subseteq D$ is a *component* of D if for all $i \in N(d)$ and $j \in N(d)$ there is a path $P_{ij}(D)$ connecting players i and j and for all $i \in N(d)$ and $j \in N(D)$, $ij \in \Delta$ implies that $ij \in d$ with d being the undirected analogue of the component d . The *set of all components* in a directed network D is denoted by $c(D) = \{d \mid d \subseteq D\}$. Note that $D = \cup_{d \in c(D)} d$.

Last, we describe the preferences of players over her potential activities. Players have complete and transitive preferences over the players in their neighborhood given by the set of potential links D and over the autarkic relation. These preferences can be represented by a hedonic utility function. The hedonic utility function is an *indirect* utility function that

captures the utility of the value generating activities. For instance, in the application of collective production the hedonic utility function is an indirect utility function that measures the utility of free time a player has when participating in collective production with another player. Let (N, D) be a directed activity structure. For every $i \in N$, there is a hedonic utility function $u_i: L_i(D) \cup \{ii\} \rightarrow \mathbb{R}$.⁸ Let u be a profile of n hedonic utility functions. Let \mathcal{U} be the set of permissible profiles of hedonic utility functions.

Below we present an example of collective production as outlined in Section 2. First we derive the hedonic utility profile in the state of autarky and then we derive individuals hedonic utility functions on the set of potential links. In the first case, autarky is a positive value generation state, hence, labor specialization and collective production do not occur between two or more distinct individuals because there are no conditions to develop social trust.

Example 3.2 The set of players is given by $N = \{1, 2, 3, 4\}$ individuals who engage in subsistence activities. Consider the collective production process described in Section 2.

Given the preference profile and the production possibility set each individual solves the following maximization problem.

$$\begin{aligned} \max \quad & \phi(f) = f = 1 - l_x - l_y \\ & 0 \leq l_x \leq 1; \\ & 0 \leq l_y \leq 1; \end{aligned}$$

subject to

$$\begin{aligned} y &\geq \bar{y}, \\ y &= x l_y, \\ x &= l_x^\alpha \\ l_x + l_y &\leq 1. \end{aligned}$$

The individual problem can be rewritten in terms of minimizing time spent in production:

$$\begin{aligned} \min \quad & l_x + l_y \\ & 0 \leq l_x \leq 1; \\ & 0 \leq l_y \leq 1; \end{aligned}$$

subject to

$$\begin{aligned} l_x^\alpha l_y &= \bar{y}, \\ l_x + l_y &\leq 1. \end{aligned}$$

Assuming the existence of a solution, the solution to the problem under autarky is given by $l_x(ii) = (\alpha \bar{y})^{\frac{1}{\alpha+1}}$ and $l_y(ii) = (\alpha^{-\alpha} \bar{y})^{\frac{1}{\alpha+1}}$, which yields $\phi(ii) = 1 - \bar{y}^{\frac{1}{\alpha+1}} \left(\alpha^{\frac{1}{\alpha+1}} + \alpha^{\frac{-\alpha}{\alpha+1}} \right)$ for all players $i \in N$.

⁸Note that we do not require that a player has a higher utility from a link in which she has the authority than from a link in which another player has authority over her. Our goal is to keep the basic framework as general as possible.

Finally we set the hedonic utility in the state of autarky equal to the utility of free time when a player is self-supplying both the intermediate and the consumption good. So, $u_i(ii) = \phi(ii)$ for all players $i \in N$.

We do not consider any other potential activities in this example, since as discussed above any collective activity between two distinct players requires trust, which only develops under extreme circumstances of necessity to survive. \blacklozenge

Example 3.2 describes the behavior of players who under autarky can sustain the minimal subsistence level \bar{y} . That is, it is assumed that the individual maximization problem has an interior solution and thus every individual in a state of autarky is self-sufficient. This assumption is violated when the minimal subsistence level cannot be met by an individual producing alone, *i.e.*, when $\bar{y} > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}$. Next we consider this case when players need to engage in collective production, hence, specialize in order to achieve the subsistence level.

Example 3.3 Consider the economy described in Example 3.2. Furthermore assume $\bar{y} > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}$. Hence, players cannot achieve the subsistence level of consumption \bar{y} in a state of autarky; however, two players producing together can obtain two times the subsistence level $\bar{y} \leq \frac{1}{2}$.⁹ In order to achieve the minimum required level of consumption they need to exploit the increasing returns to scale in the production of the intermediate goods by having some players specialize in the production of the intermediate good while others in the production of consumption good. For convenience, we will refer to player specialized in the production of the intermediate good as “he” and those specialized in the production of the final good as “she”. We assume that players 1 and 3 specialize in the production of the intermediate good while players 2 and 4 specialize in the production of the consumption good.

Consider the set of potential links $D = \{(1, 4), (4, 3), (2, 1), (3, 2), ii_{i \in N}\}$. Note that the set of potential links does not contain links between players who have the same specialization in production, *e.g.*, between two producer of the intermediate good, because such links are not value generating. The direction of a link in the set of potential links reflects “authority”. A player who has authority over another player has decision power over that player’s productive time. In return for the labor hours, the player who has authority over the other player provides her or him with the subsistence level of consumption.

We can distinguish between two cases. The first case is when a producer of the final consumption good has authority over a producer of an intermediate good. The optimization problem for the producer of the consumption good in which she chooses the amount of labor that the producer of the intermediate good should invests as well as her own labor investment is given as:

$$\begin{aligned} \max \quad & \phi(f) = f = 1 - l_y \\ & 0 \leq l_x \leq 1; \\ & 0 \leq l_y \leq 1; \end{aligned}$$

subject to

$$\begin{aligned} y & \geq 2\bar{y}, \\ y & = x l_y, \\ x & = l_x^\alpha. \end{aligned}$$

⁹Note that the maximum of the function $l_x^\alpha(1 - l_x)$ for $l_x \in [0, 1]$ is obtained at the point $l_x = \frac{\alpha}{1+\alpha}$.

Since the player producing the consumption good has preferences only over her own free time and complete decision rights over the labor hours worked by the intermediate good producer, she will require the producer of the intermediate good to allocate all his endowment of time in the production of the intermediate good $l_x = 1$ and will herself contribute just as much as it is sufficient to produce the subsistence level for both players, namely, $l_y = 2\bar{y}$. This is the solution to the collective production by potential links (2, 1) and (4, 3). Hence, $l_{x,1}(2, 1) = l_{x,3}(4, 3) = 1$, $l_{y,2}(2, 1) = l_{y,4}(4, 3) = 2\bar{y}$ where $l_{z,i}$ denotes the amount of time player $i \in N$ invests in the production of the good $z \in \{x, y\}$. The utility levels from these links of each player are thus $\phi_1((2, 1)) = \phi_3((4, 3)) = 0$ and $\phi_2((2, 1)) = \phi_4((4, 3)) = 1 - 2\bar{y}$.

The second case is when a producer of the intermediate good has a decision power over the hours worked by the player specialized in the production of the consumption good. In return for the labor hours worked by the producer of the consumption good, the producer of the intermediate good provides her with an amount of the consumption good equal to the subsistence level \bar{y} . The optimization problem of the producer of the intermediate good in which he chooses the amount of labor that the producer of the consumption good should invest as well as his own labor investment is given as:

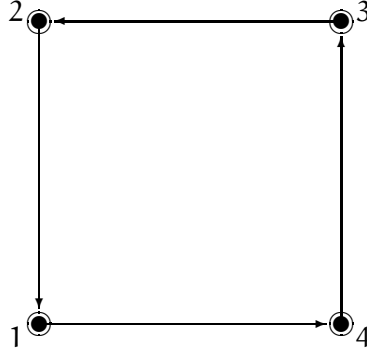
$$\begin{aligned} \max \quad & \phi(f) = f = 1 - l_x \\ & 0 \leq l_x \leq 1; \\ & 0 \leq l_y \leq 1; \end{aligned}$$

subject to

$$\begin{aligned} y & \geq 2\bar{y}, \\ y & = x l_y, \\ x & = l_x^\alpha. \end{aligned}$$

Since the player producing the intermediate good has preferences only over his own free time and complete decision rights over the labor hours worked by the consumption good producer, he will require from the producer of the consumption good to allocate all her endowment of time into production, and he will himself contribute just as much as it is sufficient to produce the subsistence level for both players, namely, $l_x = (2\bar{y})^{\frac{1}{\alpha}}$. This is the solution to the collective production by potential links (1, 4) and (3, 2). Hence, $l_{x,1}(1, 4) = l_{x,2}(3, 2) = (2\bar{y})^{\frac{1}{\alpha}}$, $l_{y,2}(3, 2) = l_{y,3}(1, 4) = 1$. The utility levels of each player from these links are thus $\phi_1((1, 4)) = \phi_3((3, 2)) = 1 - (2\bar{y})^{\frac{1}{\alpha}}$ and $\phi_2((3, 2)) = \phi_4((1, 4)) = 0$.

Due to the increasing returns to scale in the production of the intermediate good, it is easy to see that the first case is Pareto dominating the second case for $\bar{y} < \frac{1}{2}$ and the two cases yield the same total utility when $\bar{y} = \frac{1}{2}$.



Given the assumed specialization and the direction of authority, one can derive the hedonic utility function such that $u_i((i, j)) = \phi_i((i, j))$ and $u_j((i, j)) = \phi_j((i, j))$ for any link $(i, j) \in D$ and any two distinct players $i, j \in N$. Since in the state of autarky a player cannot reach a subsistence level we will denote this state as $-\infty$.

(i, j)	$i \in N$	(1,4)	(2,1)	(3,2)	(4,3)
$u_1((i, j))$	$-\infty$	$1 - (2\bar{y})^{\frac{1}{\alpha}}$	0	-	-
$u_2((i, j))$	$-\infty$	-	$1 - 2\bar{y}$	0	-
$u_3((i, j))$	$-\infty$	-	-	$1 - (2\bar{y})^{\frac{1}{\alpha}}$	0
$u_4((i, j))$	$-\infty$	0	-	-	$1 - 2\bar{y}$

Note that the way we have constructed the example, player 1 prefers to engage in production with player 4, player 2 prefers to engage in production with player 1, player 3 prefers to engage in production with player 2, and player 4 prefers to engage in production with player 3. \blacklozenge

In Example 3.3 we have derived the hedonic utility functions of players. Our goal is to develop a framework in which collective production is carried out in teams of two or more individuals. This framework is developed in the following section.

4 A Team Economy

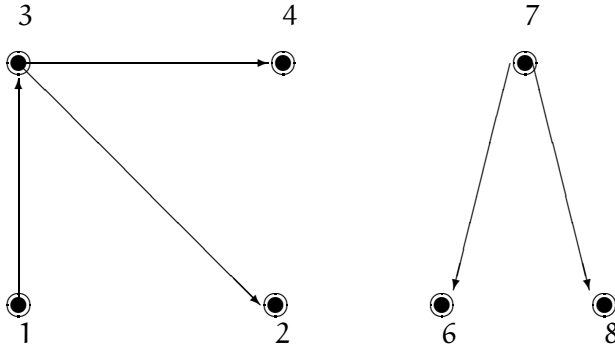
The set of potential links and hedonic utility functions described in the previous section are used as a basis for developing a framework where more complex activity patterns can be studied. In particular, the complex activity patterns that we study are represented by networks consisting of star components.

Definition 4.1 Let (N, D) be an activity structure. We say that the subset $S^* \subseteq D$ has a **star structure** or that it is a **star** if there is at most one player $i \in N(S^*)$ such that $|N_i(S^*)| > 1$ and that for all $j \in N(S^*) \setminus \{i\}$ it holds that $N_j(S^*) = \{i\}$.

Let (N, D) be an activity structure. We denote by $\mathcal{S}(D)$ the set of all subsets of D that have a star structure, i.e., $\mathcal{S}(D) = \{S^* \subseteq D \mid S^* \text{ is a star}\}$. The definition of a star includes subsets in which there are only two connected players, hence, $\{(i, j)\}_{(i, j) \in D} \subseteq \mathcal{S}(D)$.

A graphical representation of a set of potential links which contains two components with star structure is given in Figure 2. As shown, in the general framework, it may be that the

Figure 2: A Set of Potential Links with Star Components



player in the star component who has multiple links has authority in some links while other players have authority over her in other links. For example, in the pattern $\{(1, 3), (3, 2), (3, 4)\}$ player 3 has multiple links and she has authority over players 2 and 4, however, player 1 has authority over her.

Complex activities are represented by a subset of the set of potential links whose components have a star structure. To such activities we refer as star patterns.

Definition 4.2 Let (N, D) be a directed activity structure. A **star pattern** $H^* \subseteq D$ is a subset of the set of potential links such that each player $i \in N$ is either connected in a component of a star structure of H^* or stays autarkic, i.e., $c(H^*) \subseteq \mathcal{S}(D)$.

The class of all possible star patterns on D is denoted by $\mathcal{H}^*(D) = \{H^* \mid c(H^*) \subseteq \mathcal{S}(D)\}$.

Let (N, D) be an activity structure and let H^* be a star pattern. We define a *star central player* to be a player $i \in N(d^*)$ for some $d^* \in c(H^*)$ such that $|N_i(d^*)| > 1$ if $|N(d^*)| > 2$ and all players $i \in N(d^*)$ if $|N(d^*)| = 2$. The *set of star central players* in a star pattern H^* is denoted by $N^*(H^*)$.

Last, we discuss the preferences of players defined on the possible star patterns. To do so, we use an indirect value function, which is based on a player's hedonic preferences over potential links represented by her indirect utility function u_i for some player $i \in N$. In a star pattern a player can be linked to another player either directly, by activating their potential link, or indirectly via the star central player in the component. This is reflected in the *hedonic team utility function* to which we will refer as *team utility function* for brevity. Let $\mathcal{S}_i(D)$ be the set of all possible subsets of the set of potential links D in which player i is connected, then the team utility function of any player $i \in N$ is given by $v_i: \mathcal{S}_i(D) \cup \{ii\} \rightarrow \mathbb{R}$ such that for some subset $S_i \in \mathcal{S}_i(D) \cup \{ii\}$:

$$v_i(S_i) \begin{cases} = u_i(ii) & \text{if } S_i = ii; \\ = u_i((i, j)) & \text{if } (i, j) \in S_i \text{ and } |N_i(S_i)| = 1; \\ = u_i((j, i)) & \text{if } (j, i) \in S_i \text{ and } |N_i(S_i)| = 1; \\ \geq \sum_{j: (i, j) \in S_i} u_i((i, j)) + \sum_{j: (j, i) \in S_i} u_i((j, i)) & \text{if } |N_i(S_i)| > 1. \end{cases}$$

The team utility function captures the value generating abilities of a player being autarkic or being connected in a star component. The underlying value generation process may take various forms. The collective production process analyzed in the application is one example of such process. The team utility function as assumed requires that a player has value only from links with players with whom she is linked *directly*.¹⁰ Furthermore, it is allowed that a star central player receives some extra value above the sum of her utility from the relations with players in her neighborhood. Hence the value function satisfies a *superadditivity property*, i.e., $v_i(S_i \cup T_i) \geq v_i(S_i) + v_i(T_i)$ for all $S_i, T_i \in \mathcal{S}_i(D)$ with $S_i \cap T_i = \emptyset$. The superadditivity property reflects synergies which are assumed to be allocated to the star central player who acts as a coordinator in the value generation process.

The profile of team utility functions for all players is denoted by v . The set of all permissible profiles of team utility functions defined on the activity structure (N, D) is denoted by $\mathcal{V}(D)$. In a star pattern each player is engaged in a relational activity. For ease of notation we define the *indirect* team utility level that a player obtains when participating in a pattern H^* as $v_i(H^*)$. For a given star pattern the indirect team utility levels of all players are summarized in a *team utility profile* $v(H^*) = (v_1(H^*), \dots, v_n(H^*))$.

Next, we define an economy in which activity patterns of a star structure are analyzed in terms of their stability properties.

Definition 4.3 A *team economy* is defined to be a quadruple $\mathbb{E}^T = (N, D, u, v)$ in which (N, D) is a directed structure, $u \in \mathcal{U}(D)$ is a profile of utility functions on D and $v \in \mathcal{V}(D)$ is a profile of team utility functions on D .

A team economy is defined to be the set of potential actions and the potential value that a player can obtain in activating one of her possible activity patterns. A star pattern defined on this economy represents a *de facto* activated activity pattern by all individuals. We analyse a team economy in terms of the stability properties of the possible activity patterns.

Definition 4.4 Let $\mathbb{E}^T = (N, D, u, v)$ be a team economy. Let $H^* \in \mathcal{H}^*(D)$. We say that the star pattern H^* is **stable** for the economy (N, D, u, v) if it satisfies the individual rationality [IR] and two no blocking [NB] and [NB*] conditions as specified below:

IR for all $i \in N$ it holds that $v_i(H^*) \geq u_i(ii)$;

NB for all $i, j \in N$ with $(i, j) \in D \setminus H^*$ and $i, j \notin N^*(H^*)$:

$$v_i(H^*) < u_i((i, j)) \quad \text{implies} \quad u_j((i, j)) \leq v_j(H^*);$$

NB* for all distinct players $i, j \in N$ with $(i, j) \in D \setminus H^*$

- and $j \in N^*(H^*)$:

$$v_i(H^*) < u_i((i, j)) \quad \text{implies} \quad \max\{v_j(H^* \cup \{(i, j)\}), u_j((i, j))\} \leq v_j(H^*);$$

¹⁰ Alternatively, it may be that a player has utility from being linked to another player *indirectly*, e.g., $v_i(H^*) = \sum_{j \in N: j \neq i} \delta^{t(ij)-1} u_i((i, j)) + \sum_{j \in N: j \neq i} \delta^{t(ji)-1} u_i((j, i))$ where $\delta \in (0, 1)$ is the distance discounting parameter and $t(ij)$ is the geodesic distance between players i and j such that it equals the number of links in the shortest path between the players. Given our definition of a star pattern the distance between any two players can be maximum two links. If these players are not connected in the set of potential links D the geodesic distance is set to infinity. Such type of utility function has been used by Jackson and Wolinsky (1996) in a undirected network setting when discussing the *Connections Model*.

- or $i \in N^*(H^*)$:

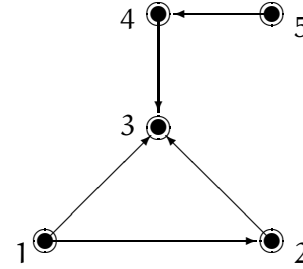
$$v_j(H^*) < u_j((i, j)) \quad \text{implies} \quad \max\{v_i(H^* \cup \{(i, j)\}), u_i((i, j))\} \leq v_i(H^*).$$

The condition [IR] is a standard condition for stability that allows players to opt out of an economic activity if she is better off being autarkic. The condition [NB] rules out blocking out possibilities between two distinct players, none of whom is a star central player in the pattern H^* . It requires that there are no pairs of players who prefer to be linked to each other rather than to the players with whom they are linked in the pattern H^* . The condition [NB*] rules out blocking possibilities between two distinct players at least one of whom is a star central player in the pattern H^* . Note the greater deviational possibilities for star central players: such players can add a links with another player with or without severing her current links. This condition requires that there are no two distinct players at least one of whom is a star central player who want to be linked to each other irrespective of whether (one of)¹¹ the star central player(s) keeps her existing links or not.

To illustrate the concepts here we provide an abstract example.

Example 4.5 Let $N = \{1, 2, 3, 4, 5\}$ and $D = \{(1, 2), (1, 3), (2, 3), (4, 3), (5, 4)\} \cup \{ii \mid i = 1, \dots, 5\}$ as shown on the figure below. Every player has utility of zero when she is in a state of autarky. The hedonic utility functions over the set of potential links between distinct players are given in the table below.

(i, j)	(1,2)	(1,3)	(2,3)	(4,3)	(5,4)
$u_1((i, j))$	2	1	–	–	–
$u_2((i, j))$	1	–	2	–	–
$u_3((i, j))$	–	1	2	1	–
$u_4((i, j))$	–	–	–	2	1
$u_5((i, j))$	–	–	–	–	2



The set of all subsets of D that have star structure is given below:

$$\begin{aligned} \mathcal{S}(D) = & \{ \{(1, 2)\}, \{(1, 3)\}, \{(2, 3)\}, \{(4, 3)\}, \{(5, 4)\}, \{(1, 2), (1, 3)\}, \{(1, 2), (2, 3)\}, \\ & \{(1, 3), (2, 3)\}, \{(1, 3), (4, 3)\}, \{(2, 3), (4, 3)\}, \{(4, 3), (5, 4)\}, \{(1, 3), (2, 3), \\ & (4, 3)\} \}. \end{aligned}$$

Thus, the class of star patterns defined on the set D is:

$$\begin{aligned} \mathcal{H}^* = & \{ \{(1, 2), \{3, 3\}, \{4, 4\}, \{5, 5\}\}; \{(1, 2), (1, 3), \{4, 4\}, \{5, 5\}\}; \{(1, 2), \{3, 3\}, \{5, 5\}\}; \\ & \{\{1, 1\}, (2, 3), \{4, 4\}, \{5, 5\}\}; \{(1, 3), (2, 3), \{4, 4\}, \{5, 5\}\}; \{(1, 2), (1, 3), \{5, 5\}\}; \\ & \{(1, 3), \{2, 2\}, \{4, 4\}, \{5, 5\}\}; \{(1, 3), \{2, 2\}, (4, 3), \{5, 5\}\}; \{(1, 3), (2, 3), \{5, 5\}\}; \\ & \{(1, 2), (2, 3), \{4, 4\}, \{5, 5\}\}; \{\{1, 1\}, \{2, 2\}, (4, 3), \{5, 5\}\}; \{(1, 2), (4, 3), \{5, 5\}\}; \\ & \{\{1, 1\}, (2, 3), (4, 3), \{5, 5\}\}; \{(1, 3), (2, 3), (4, 3), \{5, 5\}\}; \{\{1, 1\}, (2, 3), \{5, 5\}\}; \\ & \{(1, 2), (2, 3), \{5, 5\}\}; \{(1, 3), \{2, 2\}, \{5, 5\}\}; \{(1, 2), (4, 3), \{5, 5\}\}; \text{ and } \{ii\}_{i \in N}. \end{aligned}$$

¹¹Recall that in a star pattern at most one player has more than one partner in each component. Hence, it is not allowed in a star pattern to have two star central players deviating by keeping their existing links and at the same time establishing a link between each other.

For any team utility function satisfying the superadditivity property $v \in \mathcal{V}(D)$ based on the hedonic utility function u as given above. There are two stable patterns: $\{(1, 2), (2, 3), (5, 4)\}$ and $\{(1, 3), (2, 3), (4, 3), \{5, 5\}\}$. With respect to the pattern $\{(1, 2), (2, 3), (5, 4)\}$ player 4 would prefer to sever her link with player 5 and activate the link $(4, 3)$, however, player 3 prefers her current activity in the component $\{(1, 2), (2, 3)\}$ to the link $(4, 3)$. Similarly in the pattern $\{(1, 3), (2, 3), (4, 3), \{5, 5\}\}$ player 5 would prefer to be linked with 4 in the component $(5, 4)$, however, player 4 prefers her current link with player 3 to the one with player 5.

There are no other stable patterns. For instance, consider the star patterns in which players 2 and 3 are not linked. Such patterns cannot be stable because player 2 prefers to be linked to player 3 and player 3 prefers to be linked to player 2 more than to any other player. Moreover, in all other patterns in which players 2 and 3 are linked but patterns $\{(1, 2), (2, 3), (5, 4)\}$ and $\{(1, 3), (2, 3), (4, 3), \{5, 5\}\}$, the non-blocking conditions are not satisfied. Consider, for example, the pattern $H^* = \{(1, 3), (2, 3), (5, 4)\}$. It is not stable because player 4 prefers to be linked to player 5 than to player 3, $v_4(H^*) = u_4((5, 4)) < u_4((4, 3))$, and player 3 prefers to be a star central player in the component $\{(1, 3), (2, 3), (4, 3)\}$ than in the star component $\{(1, 3), (2, 3)\}$, $v_3(H^*) < v_3(H^*) + u_3((4, 3)) \leq v_3(\{(1, 3), (2, 3), (4, 3)\})$. \blacklozenge

Below, we proceed with a discussion of the team economy based on collective production.

Example 4.6 Consider the economy described in Example 3.3. Next, we present the production process when players are connected in a star pattern. First, we will derive the team utility profiles with respect to all possible star patterns. Then, we show that in this team economy, there are no stable star patterns.

First, consider the star component $\{(2, 1), (3, 2)\}$. Recall that the collective production in the link $(2, 1)$ is governed by player 2 who is a producer of the consumption good and that the link $(3, 2)$ is governed by player 3 who is a producer of the intermediate good. Furthermore, in the star component $\{(2, 1), (3, 2)\}$ three units of the subsistence level \bar{y} must be produced. The team utility function of each player is set to equal her/his utility from free time, *i.e.*, $v_i(H^*) = \phi_i(d_i)$ for $i \in N$ where $d_i \subseteq H^*$ is the component of the activity pattern H^* in which player i is linked $i \in N(d_i)$. To keep the utility levels of players 1 and 3 the same¹² as in the links $(2, 1)$ and $(3, 2)$, it is assumed that they contribute to the collective production process in $\{(2, 1), (3, 2)\}$ the same time as in the production processes of $(2, 1)$ and $(3, 2)$, respectively, *i.e.*, player 1 contributes $l_{x,1}(2, 1) = 1$ and player 3 contributes $l_{x,3}(3, 2) = (2\bar{y})^{\frac{1}{\alpha}}$. Player 2 as a star central player has to contribute the sufficient amount of labor necessary to produce three times the subsistence level \bar{y} given the amount of intermediate goods produced by players 1 and 3, which is $l_{y,2}(\{(2, 1), (3, 2)\}) = 3\bar{y} \left(1 + (2\bar{y})^{\frac{1}{\alpha}}\right)^{-\alpha}$. The value levels of each player in the star component $\{(2, 1), (3, 2)\}$ can be calculated in straightforward manner: $v_1(\{(2, 1), (3, 2)\}) = u_1((2, 1)) = 0$, $v_2(\{(2, 1), (3, 2)\}) = \phi_2(\{(2, 1), (3, 2)\}) = 1 - 3\bar{y} \left(1 + (2\bar{y})^{\frac{1}{\alpha}}\right)^{-\alpha}$ and $v_3(\{(2, 1), (3, 2)\}) = u_3((3, 2)) = 1 - (2\bar{y})^{\frac{1}{\alpha}}$. Simulations show that $v_2(\{12, 23\}) > u_2(12) + u_2(23) = u_2(12)$ for $\alpha \in [1, 10]$ and $\bar{y} \in \left(\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}, \frac{1}{2}\right]$.

Next, consider the star component $\{(2, 1), (1, 4)\}$ in which player 1 is a star central player. Recall that players 2 and 4 are producers of the consumption good and player 1 is a producer of the intermediate good. Furthermore, players 1 and 4 contribute the same amount of time in the collective production process of the pattern $\{(2, 1), (1, 4)\}$ as they did in the collective

¹²Recall that by the definition of the team utility function a player does not have utility from an indirect link with another player.

Table 1: Team Utility Profiles in Example 4.6

H^*	$v_1(H^*)$	$v_2(H^*)$	$v_3(H^*)$	$v_4(H^*)$
$\bar{i}_{i \in N}$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\{(2, 1), \{3, 3\}, \{4, 4\}\}$	0	$1 - 2\bar{y}$	$-\infty$	$-\infty$
$\{(1, 1), \{2, 2\}, \{4, 3\}\}$	$-\infty$	$-\infty$	0	$1 - 2\bar{y}$
$\{(2, 1), \{4, 3\}\}$	0	$1 - 2\bar{y}$	0	$1 - 2\bar{y}$
$\{(1, 4), \{2, 2\}, \{3, 3\}\}$	$1 - (2\bar{y})^{\frac{1}{\alpha}}$	$-\infty$	$-\infty$	0
$\{(1, 1), \{3, 2\}, \{4, 4\}\}$	$-\infty$	0	$1 - (2\bar{y})^{\frac{1}{\alpha}}$	$-\infty$
$\{(1, 4), \{3, 2\}\}$	$1 - (2\bar{y})^{\frac{1}{\alpha}}$	0	$1 - (2\bar{y})^{\frac{1}{\alpha}}$	0
$\{(2, 1), \{1, 4\}, \{3, 3\}\}$	$1 - \left(\frac{3\bar{y}}{1+2\bar{y}}\right)^{\frac{1}{\alpha}}$	$1 - 2\bar{y}$	$-\infty$	0
$\{(2, 1), \{3, 2\}, \{4, 4\}\}$	0	$1 - \frac{3\bar{y}}{(1+(2\bar{y})^{\frac{1}{\alpha}})^{\alpha}}$	$1 - (2\bar{y})^{\frac{1}{\alpha}}$	$-\infty$
$\{(1, 1), \{3, 2\}, \{4, 3\}\}$	$-\infty$	0	$1 - \left(\frac{3\bar{y}}{1+2\bar{y}}\right)^{\frac{1}{\alpha}}$	$1 - 2\bar{y}$
$\{(1, 4), \{2, 2\}, \{4, 3\}\}$	$1 - (2\bar{y})^{\frac{1}{\alpha}}$	$-\infty$	0	$1 - \frac{3\bar{y}}{(1+(2\bar{y})^{\frac{1}{\alpha}})^{\alpha}}$

productions (2, 1) and (1, 4), namely, $l_{y,2}(\{(2, 1), (1, 4)\}) = 2\bar{y}$ and $l_{y,4}(\{(2, 1), (1, 4)\}) = 1$. The star central player, player 1, contributes the minimum amount of labor in the production of the intermediate good such that given the amount of labor contributed by players 2 and 4, they can produce three times the subsistence level \bar{y} . Player 1 hence allocates $l_{x,1}(\{(2, 1), (1, 4)\}) = \left(\frac{3\bar{y}}{1+2\bar{y}}\right)^{\frac{1}{\alpha}}$ hours in the production of the intermediate good. The team utility levels of players in the pattern $\{(2, 1), (1, 4)\}$ are $v_1(\{(2, 1), (1, 4)\}) = 1 - \left(\frac{3\bar{y}}{1+2\bar{y}}\right)^{\frac{1}{\alpha}}$, $v_2(\{(2, 1), (1, 4)\}) = u_2((2, 1)) = 2\bar{y}$ and $v_4(\{(2, 1), (1, 4)\}) = u_4((1, 4)) = 0$. Note that for $\bar{y} > \frac{1}{4}$, $u_1(\{(2, 1), (1, 4)\}) > u_1((2, 1)) + u_1((1, 4)) = u_1((1, 4))$.

Given the symmetry of the set of potential links, the allocation of labor in the component $\{(3, 2), \{4, 3\}\}$ will be analogous to the one in the component $\{(2, 1), (1, 4)\}$, and the allocation of labor in the component $\{(1, 4), \{4, 3\}\}$ will be analogous to the one in the component $\{(2, 1), \{3, 2\}\}$.

The team utility profiles of the star patterns in which players are linked in pairs have been calculated in Example 3.3 as these in fact are based on the hedonic utility functions. Note that in a star pattern H^* for a player i who is connected to exactly one distinct player in H^* , e.g., $N_i(H^*) = \{j\}$ with $j \in N$, it holds that $v_i(H^*) = u_i(ij)$. The team utility profiles in all possible star patterns are summarized in Table 1.

We conjecture that in this example, there is no stable star pattern for a relevant range of the parameter values. For $\bar{y} \in (\frac{1}{4}, \frac{1}{2})$ and $\alpha \in [1, 10]$, we can show that the NB conditions are not satisfied. To see why this is the case, first note that for the specified range of the parameters a player's most preferred pattern is one in which she or he is the only star central player. Furthermore, every player prefers to be linked with another player over being autarkic. Next, consider the pattern $\{(2, 1), \{3, 4\}\}$. It is not stable because for players 1 and 4 the no-blocking condition [NB*] is not satisfied, i.e., $v_1(\{(1, 4), \{4, 3\}\}) > v_1((2, 1))$ and $v_4(\{(1, 4), \{4, 3\}\}) > v_4((4, 3))$. Furthermore, consider the star pattern $\{(1, 4), \{2, 2\}, \{4, 3\}\}$. It is not stable because players 2 and 3 do not satisfy the no-blocking condition [NB], i.e.,

$v_2((2,3)) > v_2(\{2,2\})$ and $v_3((3,2)) > v_3(\{(1,4), (4,3)\})$. Similarly, one can show that no other pattern is stable. \blacklozenge

5 Existence of Stable Patterns

In Example 4.6 we have illustrated a situation in which there is no stable star pattern. This raises the question whether there is a sufficient restriction on the set of potential links that ensures the existence of stable pattern in a given team economy. The answer for the general case is given in the following theorem and for the application concerning collective production it is given in Proposition 5.5.

Theorem 5.1 *Let (N, D, u, v) be a team economy. There exists a stable star pattern in (N, D, u, v) if the set of potential activities D does not contain a cycle or if it contains a cycle with a number of connected players $m - 1 = 3s$ with $s \in \{1, 2, \dots\}$.*

Note that the sufficient condition of existence does not take into account the direction of the cycle, *i.e.*, it requires the absence of *any* cycles; not only *directed* ones. This is because in our general framework we have not assumed any relation between the direction of a link and the hedonic utility that a player has from this link. As we will see in Proposition 5.5 when we discuss the sufficient conditions for the existence of a stable pattern in our application, it is the presence of *directed cycles* that matters. This is because in collective production there is a direct relation between the direction of a link and the utility of the players connected by this link.

Before proceeding to the presentation of a constructive proof of Theorem 5.1, we need a supplementary result. Let (N, D) be an activity structure and let $u \in \mathcal{U}$ be a profile of hedonic utility functions, we denote by $B_i(N, D, u) = \{j \in N \text{ with } ij \in \Delta \mid u_i(ij) \geq u_i(ik) \text{ for all } k \in N \text{ such that } ik \in \Delta\}$ the *set of most preferred partners* of player i for all $i \in N$ where Δ is the undirected network underlying the set of potential links D .

Lemma 5.2 *Let (N, D, u, v) be a team economy and let the set of potential links D does not contain a cycle. Then there is a pair of players $i, j \in N$ such that $j \in B_i(N, D, u)$ and $i \in B_j(N, D, u)$.*

Proof. Suppose not. Hence, for all players $i, j \in N$ such that $i \in B_j(N, D, u)$ it holds that $j \notin B_i(N, D, u)$. Consider player $i \in N$ and without loss of generality assume $B_i(N, D, u) = \{j\}$, it must be that $j \neq i$. Next, consider the set of most preferred partners of player j . Without loss of generality assume $B_j(N, D, u) = \{k\}$. It must be that $k \notin \{i, j\}$. Next, consider the set of most preferred partners of player k . Without loss of generality assume $B_k(N, D, u) = \{l\}$. It must be that $l \notin \{j, k\}$, moreover $l \neq i$ otherwise D contains a cycle. Hence, $l \notin \{i, j, k\}$. By continuing in a similar fashion, given that the player set N is finite, we establish a contradiction. \blacksquare

Proof of Theorem 5.1. Let $\mathbb{E}^T = (N, D, u, v)$ be a team economy. We will consider two cases. First we study sets of potential links that do not contain a cycle. Then we study sets of potential links that contain cycles with a number of connected players equal to $3s$ with $s \in \{2, 3, \dots\}$. We will slightly abuse notation and we will use the nondirect equivalent for a link in constructing the stable star pattern.

Case I Let the set of potential links D not contain a cycle. We will construct a star pattern that satisfies the stability conditions.

Step I: Consider a set of mutually disjoint links $\{i, j\} \in D$ such that $i \in B_j(N, D, u)$ and $j \in B_i(N, D, u)$, and denote this set by $H' \subseteq D$. Note that by Lemma 5.2 $H' \neq \emptyset$. Moreover, no players linked in H' would want to delete their link to form a link with another player. Let $M' = \{i \in N \mid |N_i(H')| = 1\}$ be the set of all players who are linked to another distinct player in H' . The player set M' contains those players who are linked in H' and who may have in the future more than one link in a stable star pattern.

Step II: Next, consider a player i who is not linked in H' such that $B_i(N, D, u) \cap M' \neq \emptyset$. If there exists a player $j \in B_i(N, D, u) \cap M'$ such that $u_j(ij) > 0$, construct $H'' = H' \cup \{ij\}$ and $M'' = M' \setminus N_j(H')$. Note that since v satisfies the superadditivity property for each player i , $u_j(ij) > 0$ is sufficient to ensure that $v_j(H' \cup \{ij\}) > v_j(H')$. Players linked in H'' will not want to delete their links and players in M'' may have more than one link in a star pattern. If there is no player $j \in B_i(N, D, u) \cap M'$ such that $u_j(ij) > 0$, choose another player k who is not linked in H such that $B_k(N, D, u) \cap M' \neq \emptyset$.

Continue in the same fashion until there is no player i who is not matched in H^ν and $B_i(N, D, u) \cap M^\nu \neq \emptyset$ and there exists a player $j \in B_i(N, D, u) \cap M^\nu$ such that $u_j(ij) > 0$ where ν is the index of the preceding iterations. That is after the last iteration which is a finite number since N is finite, the only players not linked in H^ν are (1) these players whose best partners are not linked in H^ν and (2) these players some of whose best partners are linked in H^ν but the partners linked in H^ν do not want to add a link with them.

Step III: If there is no player i who is not linked by the last iteration in H^ν and $B_i(N, D, u) \cap M^\nu \neq \emptyset$ and there is a player $j \in B_i(N, D, u) \cap M^\nu$ such that $u_j(ij) > 0$, consider all disjoint links $\{i, j\} \in N \setminus (N(H^\nu) \cup \{\{ii\} \mid ii \in H^\nu\})$ who are not linked in H^ν such that player $i \in B_j(N \setminus (N(H^\nu) \cup \{\{ii\} \mid ii \in H^\nu\}), D, u)$ and player $j \in B_i(N \setminus (N(H^\nu) \cup \{\{ii\} \mid ii \in H^\nu\}), D, u)$ and denote this set by $G \subseteq D \setminus H^\nu$. Note that by Lemma 5.2, $G \neq \emptyset$. Moreover, no players linked in $\tilde{H} = H^\nu \cup G$ would want to delete a link or can form a blocking pair with another player. Let $T = \{i \in N \mid |N_i(G)| = 1\}$ be the set of all players who are linked to another distinct player in G . Let $\tilde{M} = M^\nu \cup T$. The player set \tilde{M} contains those players who are linked in \tilde{H} who may have more than one link in a stable star pattern.

Step IV: Set $H' = \tilde{H}$ and $M' = \tilde{M}$. Go back to step II. Continue in the same fashion until there are no more players in step III. The process is finite since the player set N is finite.

Thus constructed the star pattern H is stable: players are linked to their most preferred partners out of the set of players who are also willing to be linked with them.

Case II Let the set of potential links D contain a cycle $C = (i_1, \dots, i_m)$ with $i_{k-1}i_k \in \Delta$ where Δ is the nondirected equivalent of the set D and $m \geq 7$ with $m - 1 = 3s$. Depending on the profile of team utility functions, we will distinguish two sub-cases.

Case II.1 First, consider a profile of team utility functions $v \in \mathcal{V}(D)$, such that (i) either there are two consecutive players along the cycle's path $i_{k-1}, i_k \in C$ for some $k = 1, \dots, m - 1$ with $i_0 = i_{m-1}$ such that $i_{k-1} \in B_{i_k}(N, D, u)$ and $i_k \in B_{i_{k-1}}(N, D, u)$, (ii) or there is a pair of players one of whom is on cycle path and the other other not, i.e., $i_k \in C$ ¹³ for some $k = 1, \dots, m - 1$ and $j \notin C$ such that $j \in B_{i_k}(N, D, u)$ and $i_k \in B_j(N, D, u)$. Then, we can use the algorithm described in Case I for constructing a stable star pattern since the profile

¹³By slightly abusing notation here C denotes the set of players in the sequence of players connected in the cycle.

of team utility functions ensures that the sets H' and G constructed in Step I and Step III, respectively, are not empty given the profile of team utility functions.

Case II.2 Last, consider a profile of team utility functions $v \in \mathcal{V}(D)$ such that there are no consecutive players along the cycle path $i_{k-1}, i_k \in C$ for some $k = 1, \dots, m-1$ with $i_0 = i_{m-1}$ such that $i_{k-1} \in B_{i_k}(N, D, u)$ and $i_k \in B_{i_{k-1}}(N, D, u)$, nor is there a pair of players one of whom is on cycle path and the other not, *i.e.*, $i_k \in C$ for some $k = 1, \dots, m-1$ and $j \notin C$ such that $i_j \in B_{i_k}(N, D, u)$ and $i_k \in B_j(N, D, u)$. Then, without loss of generality¹⁴, we can assume that $u_{i_k}(i_k i_k) \leq u_{i_k}(i_{k-1} i_k) < u_{i_k}(i_k, i_{k+1})$ for all $k = 1, \dots, m-1$ with $i_0 = i_{m-1}$.

A star pattern H^* that contains s number of components of the following type $\{\{i_1 i_2, i_2 i_3\}, \{i_4 i_5, i_5 i_6\}, \dots, \{i_{m-3} i_{m-2}, i_{m-2}, i_{m-1}\}\} \subseteq H^*$ and all other players are linked following the algorithm presented in Case I is stable. All players who are not linked to their most preferred partner have their most preferred partner linked to her own most preferred partner, hence, they will not sever their links, moreover, these players are not star central players, hence, they cannot add a link without severing an existing link. ■

Example 5.3 Consider the economy discussed in Examples 3.3 and 4.6. We modify the economy described in these examples in such a way that a stable star pattern exists. For this purpose we modify the direction of authority in the set of potential links as $D = \{(1, 4), (2, 1), (2, 3), (3, 4)\}$.

In this setting the collective production processes in the matchings $(2, 1)$ and $(2, 3)$ are symmetric where player 2 has a decision power over the hours worked by players 1 and 3, respectively, and so are the collective production processes in the links $(1, 4)$ and $(3, 4)$, where, respectively, players 1 and 3 have decision power over the hours worked by player 4. These problems have already been analyzed in Example 3.3, hence, we know that $l_{x,1}(2, 1) = l_{x,3}(2, 3) = l_{y,4}(1, 4) = l_{y,4}(3, 4) = 1$, $l_{x,1}(1, 4) = l_{x,3}(3, 4) = (2\bar{y})^{\frac{1}{\alpha}}$, and $l_{y,2}(2, 1) = l_{y,2}(2, 3) = 2\bar{y}$.

Next, we discuss the collective production processes in the possible star components in which more than two players are linked. First consider the star component $\{(2, 1), (2, 3)\}$. The team utility of players 1 and 3 should be the same as in the links $(2, 1)$ and $(2, 3)$, hence, $l_{x,1}\{(2, 1), (2, 3)\} = l_{x,1}(2, 1) = 1$ and $l_{x,3}\{(2, 1), (2, 3)\} = l_{x,3}(2, 3) = 1$. Since collectively these players have to produce three units of the subsistence level \bar{y} , player 2 spends the minimum hours in the production of the consumption good such that given the amount of the intermediate good produced by players 1 and 3, they produce three times the subsistence level of the of the consumption good. So, $l_{y,2}(\{(2, 1), (2, 3)\}) = 2^{-\alpha}(3\bar{y})$. The utility levels from free time can be computed in a straightforward way $\phi_1(\{(2, 1), (2, 3)\}) = \phi_3(\{(2, 1), (2, 3)\}) = 0$ and $\phi_2(\{(2, 1), (2, 3)\}) = 1 - 2^{-\alpha}(3\bar{y})$.

Next, consider the star component $\{(1, 4), (3, 4)\}$, in which both producers of the intermediate good have authority over the producer of the consumption good and a star central player, player 4. We know from the analysis of collective production in $(1, 4)$ and $(3, 4)$ that $l_{x,1}(\{(1, 4), (3, 4)\}) = l_{x,1}(1, 4) = (2\bar{y})^{\frac{1}{\alpha}}$ and $l_{x,3}(\{(1, 4), (3, 4)\}) = l_{x,3}(3, 4) = (2\bar{y})^{\frac{1}{\alpha}}$. Player 4 can utilize her position as a star central player in which she acts as a coordinator of the collective production process, and she only works as much as to produce three times the subsistence level given the amount of time contributed by players 1 and 3, *i.e.*,

¹⁴Alternatively, the profile of team utility functions $v \in \mathcal{V}(D)$ must be such that $u_{i_k}(\{i_k, i_k\}) \leq u_{i_k}(\{i_k, i_{k+1}\}) < u_{i_k}(\{i_{k-1}, i_k\})$ for all $k = 1, \dots, m-1$ with $i_0 = i_{m-1}$.

Table 2: Team Utility Profiles in Example 5.3

H^*	$v_1(H^*)$	$v_2(H^*)$	$v_3(H^*)$	$v_4(H^*)$
$\bar{i}_{i \in N}$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\{(2, 1), \{3, 3\}, \{4, 4\}\}$	0	$1 - 2\bar{y}$	$-\infty$	$-\infty$
$\{\{1, 1\}, \{2, 2\}, \{3, 4\}\}$	$-\infty$	$-\infty$	$1 - (2\bar{y})^{\frac{1}{\alpha}}$	0
$\{(2, 1), \{3, 4\}\}$	0	$1 - 2\bar{y}$	$1 - (2\bar{y})^{\frac{1}{\alpha}}$	0
$\{(1, 4), \{2, 2\}, \{3, 3\}\}$	$1 - (2\bar{y})^{\frac{1}{\alpha}}$	$-\infty$	$-\infty$	0
$\{\{1, 1\}, \{2, 3\}, \{4, 4\}\}$	$-\infty$	$1 - 2\bar{y}$	0	$-\infty$
$\{(1, 4), \{2, 3\}\}$	$1 - (2\bar{y})^{\frac{1}{\alpha}}$	$1 - 2\bar{y}$	0	0
$\{(1, 4), \{2, 1\}, \{3, 3\}\}$	$1 - \left(\frac{3\bar{y}}{1+2\bar{y}}\right)^{\frac{1}{\alpha}}$	$1 - 2\bar{y}$	$-\infty$	0
$\{(2, 1), \{2, 3\}, \{4, 4\}\}$	0	$1 - 2^{-\alpha}(3\bar{y})$	0	$-\infty$
$\{\{1, 1\}, \{2, 3\}, \{3, 4\}\}$	$-\infty$	$1 - 2\bar{y}$	$\left(\frac{3\bar{y}}{1+2\bar{y}}\right)^{\frac{1}{\alpha}}$	0
$\{(1, 4), \{2, 2\}, \{3, 4\}\}$	$1 - (2\bar{y})^{\frac{1}{\alpha}}$	$-\infty$	$1 - (2\bar{y})^{\frac{1}{\alpha}}$	$1 - \frac{3}{2^{\alpha+1}}$

$l_{y,4}(\{(1, 4), \{3, 4\}\}) = \frac{3}{2^{\alpha+1}}$. The utility levels from free time can be computed in a straightforward way $\phi_1(\{(1, 4), \{3, 4\}\}) = \phi_3(\{(1, 4), \{3, 4\}\}) = 1 - (2\bar{y})^{\frac{1}{\alpha}}$ and $\phi_4(\{(1, 4), \{3, 4\}\}) = 1 - \frac{3}{2^{\alpha+1}}$, which is higher than zero for all $\alpha \geq 1$.

Last, consider the star component $\{(1, 4), \{2, 1\}\}$. It has already been discussed in Example 4.6. Recall that players 2 and 4, respectively, contribute the following amounts of time $l_{2,y}(\{(1, 4), \{2, 1\}\}) = l_{2,y}(2, 1) = 2\bar{y}$ and $l_{4,y}(\{(1, 4), \{2, 1\}\}) = l_{4,y}(1, 4) = 1$. Player 1 contributes the minimum amount of time, so that given the hours worked by players 2 and 4, he can produce three times the subsistence level, *i.e.*, $l_{1,x}(\{(1, 4), \{2, 1\}\}) = \left(\frac{3\bar{y}}{1+2\bar{y}}\right)^{\frac{1}{\alpha}}$. The collective production in the star component $\{(2, 3), \{3, 4\}\}$ yields an analogous allocation of labor.

Having derived the utility from free time in collective production, we can compute the team utility profile in each possible star pattern. These values are presented in Table 2.

If $\bar{y} > \frac{1}{4}$, then $1 - (2\bar{y})^{\frac{1}{\alpha}} < 1 - \left(\frac{3\bar{y}}{1+2\bar{y}}\right)^{\frac{1}{\alpha}}$, players 1 and 3 will prefer to be star central players than to be linked only to player 4. For any $\alpha \geq 1$, $1 - 2\bar{y} < 1 - 2^{-\alpha}(3\bar{y})$, hence player 2 prefers to be a star central player than to be linked either to player 1 or to player 3 alone. Since $0 < 1 - \frac{3}{2^{\alpha+1}}$ player 4 prefers to be a star central player than to be linked either to player 1 or player 3 alone. Hence for $\bar{y} > \frac{1}{4}$, there are three stable star patterns: $\{(1, 4), \{2, 1\}, \{3, 3\}$, $\{\{1, 1\}, \{2, 3\}, \{3, 4\}\}$, and $\{(1, 4), \{2, 2\}, \{3, 4\}\}$. Note that in all of these patterns the player who is in a state of autarky prefers to be linked to another player, however, no player with whom she has a potential link can increase her own utility by being linked with that autarkic player compared to her utility in the component in which she is linked. \blacklozenge

In Examples 4.6 and 5.3, we illustrated that depending on the direction of the set of potential links there can be no stable star pattern or there can be multiple stable star patterns despite that the set of potential links contains a cycle. In fact in this special setting we can sharpen the sufficiency requirements for the existence of a stable star pattern in comparison to Theorem 5.1. The most important characteristics of this setting are that players are homogeneous in terms of their preference for free time; they are homogeneous within their production specialization, *i.e.*, a player can specialize to be either an intermediate good producer or a consumption good producers; any player has a higher utility from a link in which

she has authority, than from a link in which another player has authority over her or him. A team economy that satisfies these characteristics, we name a *team economy based on collective production*.

Definition 5.4 Let $\mathbb{E}^T = (N, D, u, v)$ be a team economy. Let $\{A, B\}$ be a set of roles, let $r: N \rightarrow \{A, B\}$ be a role-assignment function such that there are a non-empty set N_A of players assigned to the role A and a non-empty set N_B of players assigned to the role B with $N_A \cup N_B = N$ and $N_A \cap N_B = \emptyset$. A **team economy based on collective production** $\mathbb{E}^{TC} = (N, D^c, u^c, v^c)$ is a team economy such that the set of potential links contains only links between players of different role assignments and the autarkic states $D^c \subseteq \{N_A \otimes N_B\} \cup D_0$. All profiles of utility functions in the set of permissible hedonic profiles of utility functions $u^c \in \mathcal{U}^c(D^c)$ satisfy the following properties: there is a set of real numbers $\{\underline{u}_A, \underline{u}_B, \bar{u}_A, \bar{u}_B\}$ such that for all players $i \in N_A$ and $j \in N_B$ with $(i, j) \in D$, it holds that $u_i((i, j)) = \bar{u}_A$ and $u_j((i, j)) = \underline{u}_B$; and for all players $i \in N_A$ and $j \in N_B$ with $(j, i) \in D$, $u_j((j, i)) = \bar{u}_B$ and $u_i((j, i)) = \underline{u}_A$ with $\underline{u}_A, \underline{u}_B, \bar{u}_A, \bar{u}_B \in \mathbb{R}$.

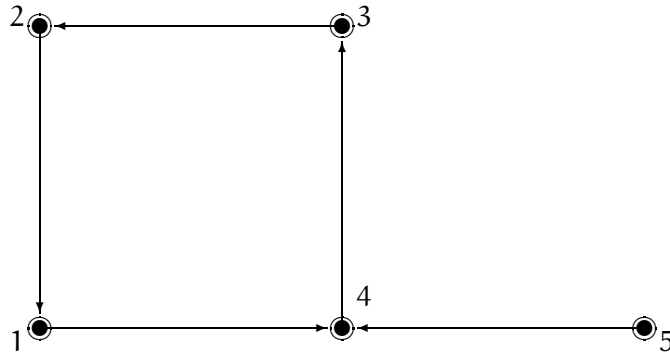
Below we present the result on existence of stable patterns in a team economy based on collective production.

Proposition 5.5 *There exists a stable star pattern in a team economy based on collective production, if the set of potential links does not contain a directed cycle.*

The proof is similar to the proof of Theorem 5.1 since the absence of directed cycles in this setting guarantees the analogous result to the result in Lemma 5.2.

The converse of Proposition 5.5, however, does not hold. That is, there are team economies of collective production in which the sets of potential links contains a directed cycle and there are stable activity patterns. Such a case is illustrated in Example 5.6.

Example 5.6 Consider the economy that has been developed in Examples 3.3 and 4.6. We modify these examples by introducing one more player who is specialized in the production of the intermediate good and who has a potential link only with player 4. The set of potential links is given by $D = \{(1, 4), (2, 1), (3, 2), (4, 3), (5, 4)\}$ and it is presented graphically below.



Due to the symmetry between the links $(1, 4)$ and $(5, 4)$, the team utility profiles of the star patterns that contain the components $d_1 = \{(5, 4)\}$ and $d_2 = \{(4, 3), (5, 4)\}$ are analogous to the team utility profiles of the star patterns containing the components $d'_1 = \{(1, 4)\}$ and $d'_2 = \{(1, 4), (4, 3)\}$ with $v_5(d_1) = v_1(d'_1)$. A new star component that we need to consider is the component $\{(1, 4), (5, 4)\}$. The same type of component in terms of the specialization of the players connected and in the direction of authority has already been analyzed in

Example 5.3, *i.e.*, the component $\{(1,4), (3,4)\}$. Hence, we know that $v_5(\{(1,4), (5,4)\}) = v_3(\{(1,4), (3,4)\}) = 1 - (2\bar{y})^{\frac{1}{\alpha}}$ and $v_4(\{(1,4), (5,4)\}) = 1 - \frac{3}{2^{\alpha+1}}$. The only new type of pattern that has not been considered so far is $\{(1,4), (4,3), (5,4)\}$. Here, by assumption of the team utility profile $v_1(\{(1,4), (4,3), (5,4)\}) = v_5(\{(1,4), (4,3), (5,4)\}) = 1 - (2\bar{y})^{\frac{1}{\alpha}}$, and $v_3 = (\{(1,4), (4,3), (5,4)\}) = v_3(4,3) = 0$. Player 4 has to contribute the minimum amount of hours in the production of the consumption good such that given the amounts of the intermediate good produced by players 1, 3 and 5, four times the subsistence level is being produced, *i.e.*, $l_{y,4}(\{(1,4), (4,3), (5,4)\}) = 4\bar{y}(1+2(2\bar{y})^{\frac{1}{\alpha}})^{-\alpha}$ and hence $v_4(\{(1,4), (4,3), (5,4)\}) = 1 - 4\bar{y}(1+2(2\bar{y})^{\frac{1}{\alpha}})^{-\alpha}$. In addition recall that the utility level in a state of autarky for each player equals $-\infty$.

Consider the star pattern $H^* = \{(1,4), (3,2), (5,4)\}$. This star pattern is stable: the only players who might want to deviate are players 2 and 4, however, the non-blocking conditions are satisfied: $u_2(2,1) = 1 - 2\bar{y} > 0 = v_2(H^*)$, however, $v_1(2,1) = 0 < 1 - (2\bar{y})^{\frac{1}{\alpha}} = v_1(H^*)$; and $v_4(H^* \cup \{(4,3)\}) = 1 - 4\bar{y}(1+2(2\bar{y})^{\frac{1}{\alpha}})^{-\alpha} > 1 - \frac{3}{2^{\alpha+1}} = v_4(\{(1,4), (5,4)\})$ for instance for $\alpha = 2$, however, $v_3(H^*) = 1 - (2\bar{y})^{\frac{1}{\alpha}} > 0 = v_3(\{(1,4), (4,3), (5,4)\})$. \blacklozenge

6 Team Generic Stability

As we discussed in Section 1, the set of potential links defines the social constraints and the social flow of authority between people, while the star pattern which emerges defines the productive patterns within a society. It is important to identify conditions which ensure the existence of stable productive patterns in any social arrangements. Such conditions will ensure what we call *team generic stability* of the activity structure (N, D) with respect to the emergence of team patterns.

Definition 6.1 *Let (N, D) be an activity structure and let $\mathcal{V}(D)$ be a set of permissible profiles of team utility functions defined on D . The activity structure (N, D) is **team generically stable** if for every $v \in \mathcal{V}(D)$ there is a stable star pattern in the team economy $\mathbb{E}^T = (N, D, u, v)$.*

The main existence theorem follows. Similarly to Theorem 5.1, the condition is based on presence of cycles and the direction of the cycle is not important.

Theorem 6.2 *The activity structure (N, D) is team generically stable if and only if the set of potential links D does not contain a cycle or if it contains a cycle, it is a cycle with $m - 1 = 3s$ and $s \in \{1, 2, \dots\}$.*

Proof. If: Let (N, D) be an activity structure and let $\mathcal{V}(D)$ be the set of permissible profiles of team utility functions defined on the set D .

The sufficiency of the conditions directly follows from Theorem 5.1 applied to every team economy (N, D, u, v) for every profile of team utility functions $v \in \mathcal{V}(D)$.

Only if: Let (N, D) be an activity structure and let $\mathcal{V}(D)$ be the set of permissible profiles of team utility function in D , and let $v \in \mathcal{V}(D)$ be a particular profile of team utility functions. We will show the necessity of the condition that D contains no cycles or that if it contains a cycle, it is a cycle with $m - 1 = 3s$ and $s = \{1, 2, \dots\}$, by contradiction.

So, let there be a stable star pattern in any team economy (N, D, u, v) for all $v \in \mathcal{V}(D)$ and let the set of potential links contain a cycle $C = (i_1, i_2, \dots, i_m)$ with $i_k, i_{k+1} \in \Delta$ for all $k = 1, \dots, m-1$ and $m \geq 4$ and suppose $m-1 \neq 3s$ with $s = \{2, 3, \dots\}$. Consider a profile of team utility functions $v \in \mathcal{V}(D)$ such that the hedonic utility profile u is given by: $u_{i_k}(i_k, j) < u_{i_k}(i_k, i_k) < u_{i_k}(i_{k-1}, i_k) < u_{i_k}(i_k, i_{k+1})$ for all $k = 1, \dots, m-1$ with $i_0 = i_{m-1}$ and all $j \in N_{i_k}(D) \setminus \{i_{k-1}, i_{k+1}\}$. Let H^* be a stable star pattern in this economy. Note that in the stable star pattern H^* the largest number of players connected in the cycle C that form a component in a star pattern, is three. We will consider two cases.

First, suppose that $i_k i_k \in H^*$ for some $k = 1, \dots, m-1$. Since H^* is a stable star pattern, the individual rationality condition is satisfied for all players in N . Hence, player i_{k-1} is in a state of autarky or connected to player i_{k-2} either in the component $g' = \{i_{k-1} i_{k-2}\}$ where she is a star central player, or in the component $g'' = \{i_{k-1} i_{k-2}, i_{k-2} i_{k-3}\}$ with $i_0 = i_{m-1}$, $i_{-1} = i_{m-2}$, and $i_{-2} = i_{m-3}$ where she is not a star central player. In all three cases one of the non-blocking conditions is violated: if player $i_{k-1} \notin N^*(H^*)$, then the condition [NB] is violated since by the construction of the profile of hedonic utility functions $u_{i_k}(i_{k-1}, i_k) > u_{i_k}(H^*)$ and $u_{i_{k-1}}(i_{k-1}, i_k) > u_{i_{k-1}}(H^*)$; in case $i_{k-1} \in N^*(H^*)$, then the condition [NB*] is violated since $u_{i_k}(i_{k-1}, i_k) > u_{i_k}(H^*)$ and $v_{i_{k-1}}(H^* \cup \{i_{k-1}, i_k\}) \geq u_{i_{k-1}}(i_{k-2}, i_{k-1}) + u_{i_{k-1}}(i_{k-1}, i_k) > u_{i_{k-1}}(H^*)$. Since H^* is stable, then it cannot be that $\{i_k, i_k\} \in H^*$ for some $i_k \in C$ ¹⁵.

Next, suppose that there is no player along the cycle's path such that $i_k i_k \in H^*$ and in addition $m-1 \geq 4$. Since H^* is a stable star pattern, the individual rationality condition is satisfied for all players in N . Since $m-1 \neq 3s$ with $s = \{2, 3, \dots\}$, $m-1 \geq 4$ and there is no player i_k along the cycle's path such that $i_k i_k \in H^*$, there must be at least two distinct players along the cycle's path i_{k-1} and i_k for some $k = 1, \dots, m-1$ and $k_0 = m-1$ such that the component $\{i_{k-1}, i_k\} \in H^*$. However, in that case, the non-blocking condition [NB*] is violated: $u_{i_{k-2}}(i_{k-2}, i_{k-1}) > u_{i_{k-2}}(H^*)$ and $v_{i_{k-1}}(\{i_{k-2} i_{k-1}, i_{k-1} i_k\}) \geq u_{i_{k-1}}(i_{k-2} i_{k-1}) + u_{i_{k-1}}(i_{k-1} i_k) > u_{i_{k-1}}(H^*)$ with $k_{-1} = m-2$ and where the first inequality follows from the superadditivity of the team utility function. ■

In team economies based on collective production, where we consider only a sub-class of utility functions \mathcal{U}^c such that the hedonic utility function is contingent on the direction of the directed links, the sufficient condition for the existence of a stable pattern requires the absence of *directed cycles*. Furthermore, as Example 5.6 showed, the absence of directed cycles is not a *necessary condition*. The necessary and sufficient condition presented in Proposition 6.4 requires that the set of potential links does not contain *components* that consist of directed cycles.

First, we define the activity structure pertaining to a team economy based on collective production.

Definition 6.3 Let (N, D) be an activity structure. Let $\{A, B\}$ be a set of roles, let $r: N \rightarrow \{A, B\}$ be a role-assignment function such that there are a non-empty set N_A of players assigned to the role A and a non-empty set N_B of players assigned to the role B with $N_A \cup N_B = N$ and $N_A \cap N_B = \emptyset$. The pair (N, D^c) is an **activity structure pertaining to an economy based on collective production** if $D^c \subseteq \{N_A \otimes N_B\} \cup D_0$.

The result follows.

¹⁵Here again we slightly abuse the notation and use C to denote the set of connected players in the cycle.

Proposition 6.4 *Let (N, D^c) be an activity structure pertaining to an economy based on collective production. Then (N, D^c) is team generically stable if and only if the set of potential links does not contain a directed cycle C^d with $m - 1 \neq 3s$ with $s \in \mathbb{N}$.*

The proof of Proposition 6.4 is very similar to the one of Theorem 6.2 and uses the results of Proposition 5.5. What should be kept in mind is that in team economies based on collective production only cycles of even number of players are possible. This follows from the restriction on the set of potential links that it can only include links between players of different role assignments.

7 Final Remarks

The results that we derive in our application to pre-market collective production, are in line with the anthropological insights that the recognition of authority is a necessary condition for the emergence of complex production. To ensure team generic stability in a team economy based on collective production a social structure must not contain components of directed cycles, which implies that there must be a well-defined flow of authority in all components of the social structure. The presence of directed cycles in a set of potential links has a very intuitive interpretation in term of the structure of an organization in which the potential links stand for value generating activities. Within such an organization a directed cycle would mean that players have authority over players who have indirect authority over them.

Note that in our application of collective production we have not assumed an ex-ante relation between a production role and authority, *i.e.*, it has not been assumed that all players specialized in the production in one of the types of goods, intermediate or consumption, have authority over the players specialized in the other type of group. Such a one-to-one relation between labor specialization and authority, however, would imply that the set of potential links does not contain a directed cycle. Hence, what some anthropologists identify as the path for the emergence of complex relations, labor specialization, in our framework is an ex-ante realization of the necessary and sufficient condition for guaranteed ex-post stability.

A one-to-one relation between production roles and authority, however, is just one possible way of satisfying the necessary and sufficient condition for stability. There are other organizations of the sets of potential links that would satisfy this condition. For instance, one such structure is a set of potential links in which producers of intermediate goods belong to either one of two types: either they have authority over all the producers of the consumption good with whom they are linked, or all the producers of the consumption good with whom they are linked have authority over them.

Moreover, we should reiterate that, according to our analysis, it is not necessary that authority emanates from a star central player. This may be the case, if the set of potential links is such that there is a one-to-one relation between production role and authority as discussed above.

We should also point out the fact that in the theoretical results applicable to the general framework, Theorems 5.1 and 6.2, the direction of a cycle does not play a role. This is because in the general setting we have not assumed a relation between a player's possession of authority and the utility she derives from a link with another player. This makes this framework applicable to a broad class of situations, also such which can be modeled by undirected graphs.

Last, we should mention some limitations of our general framework. In particular, in our work we focus on very special class of activity patterns, which consists of star structures. For complex production processes, such as hierarchies of several levels, predominant in today's economic world, these tools are inadequate. A clear goal for future work is the development of a framework where more complex patterns can be analyzed.

References

- Arnold, J. (1993), 'Labor and the rise of complex hunter-gatherers', *Journal of Anthropological Archeology* **12**, 75–119.
- Coase, R. A. (1937), 'The nature of the firm', *Economica* **4**, 386–405.
- Gilles, R. P., Lazarova, E. A. and Ruys, P. H. M. (2007), 'Stability, specialization, and social recognition', *Division of Labour and Transaction Costs* . forthcoming.
- Jackson, M. O. and Wolinsky, A. (1996), 'A strategic model of social and economic networks', *Journal of Economic Theory* **71**, 44–74.
- Kahneman, D. and Thversky, A. (1979), 'Prospect theory: an analysis of decision making under risk', *Econometrica* **47**, 263–292.
- Minnegal, M. and Dwyer, P. (1998), 'Intensification and social complexity in the interior lowlands of papua new guinea: A comparison of bedamuni and kubo', *Journal of Anthropological Archeology* **17**, 375–400.
- Pápai, S. (2004), 'Unique stability in simple coalition formation games', *Games and Economic Behavior* **48**(2), 337–354.
- Yang, X. (2003), *Economic Development and the Division of Labor*, Blackwell Publishing, Malden, Massachusetts.