Dynamic Multilateral Markets☆

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Abstract

We study dynamic multilateral markets, in which players' payoffs result from coalitional bargaining. In this setting, we establish payoff uniqueness of the stationary equilibria when players exhibit some degree of impatience. We focus on market games with different player types, and derive under mild conditions an explicit formula for each type's equilibrium payoff as market frictions vanish. The limit payoff of a type depends in an intuitive way on the supply and the demand for this type in the market, adjusted by the type-specific bargaining power. Our framework may be viewed as an alternative to the Walrasian price-setting mechanism. When we apply this methodology to the analysis of labor markets, we can determine endogenously the equilibrium firm size and remuneration scheme. We find that each worker type in a stationary market equilibrium is rewarded her marginal product, i.e. we obtain a strategic underpinning of the neoclassical wage. Interestingly, we can also replicate some standardized facts from the search-theoretical literature such as positive equilibrium unemployment.

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1. Introduction

In many economic situations, the surplus from trade depends on the actions and preferences of multiple agents. This is certainly the case in inter-connected markets like supply chains, markets where intermediaries play a prominent role, for example financial or real estates markets, and labor markets. One may argue that the standard two-sided markets models offer a good enough approximation of the more complex environment in these multilateral examples; but do they? Questions such as how prices are formed in multilateral markets; whether they are unique and anonymous; how they are related to the benchmark competitive price resulting from a tâtonnement process are the subject of this study.

Our aim in developing a framework for the study of multilateral markets is to capture a competitive environment in which strategic behavior by players leads to the formation of prices. To this end, we generalize Rubinstein and Wolinsky's (1985) seminal model of a strategic buyer-seller market to multilateral markets. We also follow these authors in employing the notion of stationary market equilibrium for determining equilibrium prices. Like Gul (1989), Chatterjee, Dutta, Ray, and Sengupta (1993), and more recently Yan (2005) we find that employing stationarity strengthens the predictive power of the equilibrium.[1] In fact, we show the payoff uniqueness of the stationary equilibria in all games in which players exhibit some degree of impatience (Theorem 1).

To analyze in greater detail the properties of equilibrium payoffs, we extend our analysis to market games with heterogeneous types of players that may be treated as type-specific factors of production. Here, we focus on the limit behavior of players, i.e. we study stationary equilibria when players become patient. In this environment, we show that players of the same type receive identical limit equilibrium payoffs, i.e. that prices are anonymous; and that these payoffs exhaust the value of those coalitions which reach an agreement (Theorem 2, items (i) and (ii)). This result implies the endogenous emergence of unique limit equilibrium price for each factor of production.

[1] The multiplicity of perfect equilibria in the extension of Rubinstein's (1982) model to multilateral bargaining has been first pointed out by Shaked and reported by Sutton (1986).

Furthermore, we establish that each factor of production in a limit market equilibrium is rewarded (approximately) its marginal contribution to the coalition where agreement is reached (Theorem 2, item (iii)). As such, unlike Gul (1989) and Hart and Mas-Collel (1996), we find that the limit market equilibria of our market game are unrelated to the Shapley value of the corresponding transferrable utility game.

The last of our general results, focuses on the limit equilibria in which the ratio of the coalitional values to the coalitional limit equilibrium payoffs is different for any two coalitions with different type profiles and which share at least one type of members. We name these equilibria separating. We show that a separating limit market equilibrium induces a partition of player types according to types of coalitions in which they reach an agreement and fully characterize players' equilibrium payoffs (Theorem 3).

The relation between the neoclassical price and the equilibrium payoff in the model becomes clearer when we apply our bargaining methodology to the study of the labor market. Here the neoclassical wage presents a natural benchmark for the remuneration schedule derived in equilibrium. Somewhat surprisingly,[2] we find that in equilibrium when workers are patient, each worker's share of the firm's value equals (approximately) her marginal contribution to the firm, i.e. her marginal product of labor. In essence, we provide a strategic underpinning of the neoclassical competitive wage.

This finding differs from those of Stole and Zwiebel (1996a,b) and Okada (2011) and is similar in spirit to that of Westermark (2003). A crucial difference between this work and that of Westermark (2003), however, is that while we study multilateral bargaining, wages are determined by a sequence of bilateral bargaining events in his model. In particular, we find that the parity between the neoclassical wage and limit payoffs in this model coexists with some standardized facts shown in the search-theoretic literature[3] such as a positive level of equilibrium unemployment even when

[2]Rubinstein and Wolinsky (1985, 1990) find that strategic bargaining may not lead to the competitive market outcome when applying a similar bargaining methodology to the study of buyer-seller markets.

[3]For a recent survey of the search-theoretic literature see Rogerson, Shimer, and Wright (2005).

players are patient.[4] In addition, we find that in equilibrium, worker's relative wage does not depend on bargaining frictions, as embodied in the discount factor and the matching probabilities. Instead, it depends in an intuitive way on the relative bargaining power between the entrepreneur and the worker and it is an increasing function of the relative labor market tightness given by the ratio of the number of vacancies and total labor supply. Importantly, our model allows for an endogenous selection of firm size (i.e., the composition of agreeing coalitions). For the Cobb-Douglas economy, our analysis shows further that the equilibrium firm size is positively related to the total labor supply and it is negatively related to the concavity of the production function.

With its focus on dynamic multilateral bargaining, this paper contributes to two growing branches of literature: that on dynamic markets (for example, in the context of seller-buyer markets see Rubinstein and Wolinsky (1985), Binmore and Herrero (1988), Rubinstein and Wolinsky (1990), and, more recently Duffie, Gâarleanu, and Pedersen (2005) and Manea (2011)); and that on coalitional bargaining (e.g. Chatterjee et al. (1993), Hart and Mas-Collel (1996), Chatterjee and Sabourian (2000), and more recently Okada (2011)). Based on its bargaining procedure, our work is closest in nature to Manea (2011). In fact, our Theorem 1 can be seen as a generalization of Manea's result on the payoff-uniqueness when the matched bilateral coalitions are network links with normalized values. We show that this uniqueness holds also in bargaining games on weighted networks and, more generally, in games on hypergraphs that represent multilateral coalitions. Our work differs from other studies on multilateral markets in the bargaining procedure and matching mechanism. Several other works study multilateral bargaining, however, they do not consider a dynamic market. Some of these focus on characteristic function form games. These authors either aim at supporting core allocations as equilibrium outcomes (cf. Yan (2003)) or focus on the efficiency of the bargaining outcomes (cf. Chatterjee et al. (1993) and Okada (1996)). In a related contribution, Krishna and Serrano (1996) allow for "partial agreement" where the agreeing agents leave the bargaining pro-

[4] Neither Stole and Zwiebel (1996a,b) or Westermark (2003) find a positive level of equilibrium unemployment in their models. Okada (2011), on the other hand, finds a possibility for a positive unemployment only if players have some degree of impatience.

cedure with their agreed shares, while the proposer and the disagreeing agents proceed to the next stage of bargaining. While the assumption of "partial agreement" may be valid in certain contexts, e.g., division of an estate, it is less applicable to others such as production with complementary inputs. Chatterjee and Sabourian (2000), instead, study unanimous agreement of all parties in the multi-person ultimatum game. They, however, assume that bargaining continues until agreement is reached. As our focus is on anonymous dynamic markets, the assumption that the bargaining coalition dissolves in case of disagreement seems more plausible. Finally, Okada (2011) studies coalitional bargaining in the context of coalition formation where a proposal is a pair of a coalition of players and a payoff vector.

The rest of the paper is organized as follows: In Section 2 we describe our theoretical framework and in Section 3 we state our main existence result. In Section 4, we specialize our discussion of market equilibria by introducing heterogeneous player types. In this context, we develop the notions of limit market equilibrium and of separating limit market equilibrium and we discuss their characterization. We employ the theoretical results to the study of the labor market with homogeneous and heterogeneous labor force in Section 5. Section 6 provides some concluding remarks.

2. Market Interaction

We consider a market with a finite set $\mathcal{N}$ of replica agents that operates over an infinite number of discrete dates. As any agent $i \in \mathcal{N}$ that leaves the market is instantly replaced by its replica, the set $\mathcal{N}$ remains in the market throughout the entire history of the game. We assume that all replicas discount future dates with a common factor $\delta \in (0, 1)$.

Each date starts with a matching stage, in which a (possibly empty) coalition of players (subsets of $\mathcal{N}$) is randomly matched. Specifically, the probability π_S of selecting the coalition $S \subseteq \mathcal{N}$ is implied by a stationary (exogenous) matching procedure and, hence, is constant throughout the game. When matched, the members of S can produce a surplus $v(S) \geq 0$ by employing their player-specific inputs. The production of $v(S)$ can only take place if all members of S agree on the terms of trade. The latter are negotiated either according to the Nash Bargaining Solution

(NBS) or one of the players is chosen as proposer in an ultimatum bargaining game.[5] In the latter case, proposer $i \in S$ makes sequentially (in any order) offers to the players in $S \backslash \{i\}$. Like similar models that adopt the alternating offer mechanism of Rubinstein (1982), cf. Chatterjee et al. (1993), we assume that the proposer is fully committed to the offers, once they are made. If an offer is rejected, coalition S dissolves immediately, i.e., without any further offers being made, and the same population of players proceeds to the next date. Otherwise, there is an agreement, in which case the value $v(S)$ is created, agents in $S \backslash \{i\}$ receive their agreed shares and i obtains the residual surplus. Then, the members of S leave the market and are replaced by their replicas with the same endowments before the game moves to the next date. Importantly, all new agents are treated by the matching procedure in the same way as the ones who left. In particular, the set of newcomers that have replaced the members of an agreeing coalition S, will be selected with probability π_S in all subsequent periods, in which it stays in the market.[6]

We parametrize the (absolute) bargaining power of player $i \in \mathcal{N}$ by α_i, which is assumed to be strictly positive and the same for all replicas of i. Let the set of all coalitions $S \subseteq \mathcal{N}$ containing player $i \in \mathcal{N}$ be denoted by S_i. Then, the probability that player i is proposer in $S \in S_i$ is given by $\alpha_i/\alpha(S)$, where $\alpha(S) := \sum_{k \in S} \alpha_k$, for all $i \in S$. In the context of Nash Bargaining, $\alpha_i/\alpha(S)$ is the (relative) bargaining power of i in coalition S. In the next section, we explain how the threat points (or minimum acceptance levels) are determined endogenously in a stationary market equilibrium.

3. Stationary Equilibria

The market interaction described in the previous section defines a game with complete but imperfect information as we assume that players do not observe the terms of agreements they are not

[5]As all coalitional members need to agree for a proposal to be carried out, the proposer makes an offer of what Huang (2002) refers to as *conditional* nature.

[6]Similar assumptions are made by Manea (2011) and Mortensen and Wright (2002) in the study of seller-buyer markets. Alternatively, we can focus on steady states in which, by definition, this condition is also satisfied.

part of.[7] One way to analyze this game is to follow, e.g., Rubinstein and Wolinsky (1985) - RW 1985 hereafter - and define its extensive form. This definition specifies histories and strategies for all players. In particular, for any agent, the history at a certain stage of the game is the sequence of observations made by her up to that stage. A strategy for any agent is, then, a sequence of decision rules conditional on all histories that dictates player's moves, i.e., the offers she makes as a proposer and her acceptance/rejection responses to offers made to her. RW 1985 focus on stationary strategies, i.e., strategies that prescribe a history-independent bargaining behavior towards the partners with whom an agent is matched. They define market equilibrium (ME thereafter) as a stationary strategy profile such that no agent can improve by changing her action after any possible history.

In this work, we consider only MEa and, therefore, we omit the formalization of histories and history-dependent strategies.[8] At any date t, a stationary strategy of proposer $i \in S$ in a matched coalition S will depend exclusively on the identities of responders in $S \backslash \{i\}$. Responder $j \in S$, on the other hand, will condition her stationary strategy on the coalition S, the identity of the proposer $i \in S$ and on the offer made to her. It follows that for stationary strategy profile, matching environment and bargaining powers, each replica of player $k \in \mathcal{N}$ expects the same payoff x_k at the beginning of any date. We exploit this fact to characterize MEa by deriving conditions for ME payoffs $\{x_k\}_{k \in \mathcal{N}}$.

Specifically, if $j \in S$ is the last responder in the matched coalition S, she will accept in equilibrium any offer larger than δx_j and rejects any offer smaller than δx_j. If the last but one responder $k \in S$ anticipates the acceptance by j, she will accept any offer larger than δx_k and rejects any offer smaller than δx_k. This argument propagates to all responders in S. Then, if

$$v(S) - \delta x(S \backslash \{i\}) > \delta x_i \Leftrightarrow v(S) > \delta x(S),$$

all responders in S accept in equilibrium their respective continuation payoffs with probability 1.

[7] This assumption is inoccuos for our results that focus on stationary equilibria.

[8] Such formalization would be a straightforward generalization of that in Manea (2011) to coalitions with more than two members.

Otherwise, proposer $i \in S$, who obtains the residual surplus $v(S) - \delta x(S\backslash\{i\}) > \delta x_i$, would be better off by increasing offers infinitesimally to ensure all sequential acceptances. If $v(S) < \delta x(S)$, on the other hand, the residual claim is less than proposer's continuation payoff δx_i and i will make at least one unacceptable offer. This will lead to disagreement in S and respective continuation payoffs for all members of S. Finally, if $v(S) = \delta x(S)$, players in S are indifferent between agreement and disagreement, which can result in a randomized equilibrium agreements in S. This analysis of ME strategies is succinctly captured by the following system of equations,

$$\begin{aligned} x_i &= \sum_{S \in \mathcal{S}_i} \pi_S \left(\frac{\alpha_i}{\alpha(S)} \max\{v(S) - \delta x(S\backslash\{i\}), \delta x_i\} + \frac{\alpha(S\backslash\{i\})}{\alpha(S)} \delta x_i \right) + (1 - \sum_{S \in \mathcal{S}_i} \pi_S) \delta x_i \\ &= \delta x_i + \sum_{S \in \mathcal{S}_i} \pi_S \frac{\alpha_i}{\alpha(S)} \max\{v(S) - \delta x(S), 0\}, \quad \forall i \in \mathcal{N}. \end{aligned} \tag{1}$$

A solution $x^\delta = (x_i^\delta)_{i \in \mathcal{N}}$ to the system (1) yields the expected ME payoffs. A unique solution x^δ, in particular, yields a unique ME payoff profile although this profile can support multiple MEa. Multiplicity occurs, for example, when responder j in the matched coalition S, for which holds $v(S) = \delta x^\delta(S)$, agrees with any probability to the offer δx_j^δ.

We will say that S is active in ME x^δ if S agrees in this equilibrium with positive probability. Further, we will say that player i cooperates in ME x^δ if $i \in S$ and S is active in x^δ. The pattern of cooperation among coalitions in a stationary equilibrium will be called coalitional cooperation structure.

Note that re-writing (1) as,

$$x_i = (1 - \textstyle\sum_{S \in S_i} \pi_S) \delta x_i + \textstyle\sum_{S \in S_i} \pi_S \max\{\delta x_i + \dfrac{\alpha_i}{\alpha(S)} (v(S) - \delta x(S)), \delta x_i\}, \tag{2}$$

makes clear that the expected payoff x_i results also when the outcome of each coalitional meeting is prescribed by the NBS, where player i's bargaining power is given by $\alpha_i/\alpha(S)$ and the (endogenous) threat points are $(\delta x_i)_{i \in S}$.

We can interpret the payoff x_i as the price that player i expects for her input in a ME. Note that x_i is different from player i's expected agreement payoff. This distinction would have had important implications, had the agents entered the market with a stock of inputs. If agents had traded repeatedly, the disagreement payoffs would have not internalized the utility loss due to delayed cooperations. Although we allow each replica agent to trade only once, our model can be easily extended in this direction.

The following Example 1 illustrates equilibrium payoffs in several stylized market settings.

Example 1. We consider a market with three players, $\mathcal{N} = \{1, 2, 3\}$, who have equal bargaining powers. Below we discuss several cases of value functions that capture various standard market environments. To simplify our discussion and notation, in each case we only list the values of the productive coalitions and take $\pi_S = 0$ for all unproductive coalitions $S \subseteq \mathcal{N}$, i.e. $v(S) = 0$. In addition, we will assume that all productive coalitions are matched with equal probabilities and that the sum of these probabilities is one.

Bilateral Bargaining First, we consider a very simple buyer-seller market discussed by Rubinstein and Wolinsky (1990) where $v(\{1, 2\}) = 1$. Then, the unique solution to system (1) yields the ME payoff of $1/2$ to each player in the coalition $\{1, 2\}$ and zero to the unproductive player 3. Clearly, this market equilibrium is independent of the discount factor δ.

When we let $v(\{1, 2\}) = v(\{1, 3\}) = 1$, we obtain a standardized example of a buyer (player 1) who bargains with two identical sellers (players 2 and 3) over the price of one unit of a homogenous good. The unique ME solution here,

$$x_1^\delta = \frac{2}{4-\delta}, \quad x_2^\delta = x_3^\delta = \frac{1}{4-\delta},$$

illustrates the fact that the single buyer cannot extract the entire surplus from the competing sellers for any level of impatience (market frictions).

Generalizing the case above to a setting with two sellers of heterogenous goods, we set $v(\{1, 2\}) = 1$ and $v(\{1, 3\}) = a \geq 1$. Then, by taking the limit $\delta \to 1$ of the solution to (1), we obtain,

$$\begin{aligned} x_1^1 &= \frac{1+a}{3}, \quad x_2^1 = \frac{2-a}{3}, \quad x_3^1 = \frac{2a-1}{3}, \quad \text{if} \quad a \leq 2, \\ x_1^1 &= a/2, \quad x_2^1 = 0, \quad x_3^1 = a/2, \quad \text{if} \quad a > 2. \end{aligned}$$

Thus, if players are patient, both buyer-seller pairs trade when $a < 2$, while only coalition $\{1, 3\}$ agrees when $a > 2$. Note that all payoffs vary continuously in a.

Multilateral bargaining As in the benchmark bilateral example before, when only the three-player coalition is productive, the outcome of the market game is to split the value of the coalition among its members irrespective of the discount factor δ. For example, $v(\{1, 2, 3\}) = 1$ models trilateral bargaining where the unique ME payoff yields $1/3$ to each player.

Alternatively, multilateral bargaining may be captured by the case where $v(S) = 1$ for all S with at least two members. The unique solution to system (1) for $\delta > 4/5$,[9]

$$x_i^\delta = \frac{1}{2(2-\delta)}, \quad i = 1, 2, 3,$$

implies an agreement by any matched pair of players and a disagreement in the grand coalition. It also shows that the ME payoffs can differ from the Shapley values of the corresponding static game and that they are not in the core of this game, which is empty.

Finally, we consider a stylized example of intermediation. We let $v(\{1, 2\}) = v(\{1, 2, 3\}) = 1$ be a market with one buyer (player 1), one seller (player 2) and an (unproductive) intermediary (player 3). The solution to (1),

$$x_1^\delta = x_2^\delta = \frac{10 - 9\delta}{2(12 - 12\delta + \delta^2)}, \quad x_3^\delta = \frac{4(1-\delta)}{2(12 - 12\delta + \delta^2)}, \tag{3}$$

illustrates, that the unique ME payoff to the buyer (seller) increases with δ, reaching $1/2$ in the limit $\delta \to 1$. The unique equilibrium payoff for the intermediary, on the other hand, is decreasing with δ, reaching 0 when $\delta \to 1$. As expected, the intermediary's profit vanishes as the market becomes frictionless.

[9]For $\delta < 4/5$ the solution $x_i^\delta = \frac{4}{3(4-\delta)}$ to (1) implies an agreement in all matched coalitions. For $\delta = 4/5$, we have that $3\delta\frac{4}{3(4-\delta)} = 3\delta\frac{1}{2(2-\delta)} = 1$ and an agreement with any probability is rational in the grand coalition.

We end this section by stating our fundamental result of existence and uniqueness.

Theorem 1. *There exists a ME in any market game. All MEa yield a unique payoff profile.*

Proof: All proofs are relegated to the Appendix.

4. Limit Market Equilibria and Multilateral Coalitions

In this section, we focus on a class of dynamic market games, where players of distinct types interact. Specifically, we consider a partitioning of the set of agents $\mathcal{N}$ into T types, i.e., $\mathcal{N} = \cup_{t=1,\ldots,T} \mathcal{N}_t$ and $\mathcal{N}_s \cap \mathcal{N}_t = \varnothing$ for all $s, t = 1, \ldots, T$ with $s \neq t$. For all types $t = 1, \ldots, t$, we denote the cardinality of the set $\mathcal{N}_t$ by N_t. A multilateral coalition (MC) $S \subseteq \mathcal{N}$ consists of $\sum_{t=1,\ldots,T} n_t$ players, where $n_t = |S \cap \mathcal{N}_t|$ denotes the number of t-type players in this coalition. For any MC $S \subseteq \mathcal{N}$, we use the operator $T(S)$ to obtain the type profile of its members. Formally, given a MC $S \subseteq \mathcal{N}$, its type $T(S)$ is defined as an ordered vector of type multiplicities, i.e., $T(S) = (n_1, ..., n_T)$ where $n_t = |S \cap \mathcal{N}_t|$ for $t = 1, ..., T$. For example, for the grand coalition we get $T(\mathcal{N}) = (N_1, ..., N_T)$ while for a singleton coalition $\{i\}$ with $i \in \mathcal{N}_t$, we obtain $T(\{i\}) = \mathbf{e}_t$, where $\mathbf{e}_t$ is a vector in the canonical basis of the Euclidean space that points in direction t. We assume that coalitions of the same type have the same productivity, i.e., $v(S) = v(S')$ when $T(S) = T(S')$. Therefore, we can use the shorthand notation $v(T(S))$ for the productivity of any coalition of type $T(S)$. We assume further that all productive coalitions have a positive probability of meeting, $\pi_S > 0$ if $v(S) > 0$. Finally, we require that players of the same type have equal absolute bargaining powers, $\alpha_i = \alpha_j$ if $T(\{i\}) = T(\{j\})$.

Similar to Manea (2011) and Chatterjee et al. (1993), we study the market outcome when participants are patient. Our aims are to characterize agreements in coalitions of different types and derive explicit expressions for the players' limit equilibrium payoffs. Formally, a limit market equilibrium is defined below.

Definition 1. A ME x^1, where $x^1 := \lim_{\delta \to 1} x^\delta$ and x^δ is a solution to (1) will be called a limit ME (LME).

Note that LME is well-defined as the solution x^δ is unique by Theorem *1*. The following result establishes payoff-equivalence between players of the same type, the value exhaustion, and the bounds on the LME payoffs.

Theorem 2. *Consider a market game and a LME x^1 of this game. Then*

(i) all players of type $t = 1, \ldots, T$, receive the same payoff x^1_t;

(ii) for an active coalition of type $\mathbf{n} = (n_1, ..., n_T)$, it holds that

$$\sum\nolimits_{s=1}^{T} n_s x^1_s = v(\mathbf{n});$$

(iii) a player of type t that cooperates in a LME in coalitions of type $\mathbf{n}$ receives (approximately) her marginal contribution to this type of coalitions,

$$v(\mathbf{n} + \mathbf{e}_t) - v(\mathbf{n}) \leq x^1_t \leq v(\mathbf{n}) - v(\mathbf{n} - \mathbf{e}_t), \tag{4}$$

whenever $v(\mathbf{n} + \mathbf{e}_t)$ and $v(\mathbf{n} - \mathbf{e}_t)$ are well-defined.[10]

Next, we introduce the notion of a separating limit market equilibrium (SLME) that allows us to derive explicit LME payoffs as a function of market fundamentals.

Definition 2. A LME x^1 is separating if,

$$\forall S, S' \subseteq \mathcal{N} : S \cap S' \neq \varnothing, T(S) \neq T(S'), \quad v(S)x^1(S') \neq v(S')x^1(S).$$

In a SLME, the ratio of the coalitional values to the coalitional LME payoff (when the latter is greater than zero) is different for any two intersecting coalitions of different types. Although only such limit equilibria will arise in many games, e.g., the standard buyer and two identical sellers case discussed in Example 1, one can easily construct examples, where the unique LME is not separating. For instance, in the intermediation game, discussed last in Example 1, the members of the buyer-seller coalition $S = \{1, 2\}$ obtain the joint limit payoff of one, which is the same as the

[10]The unit vector $\mathbf{e}_t$ belongs to the canonical basis of the Euclidean space and points in direction t.

joint limit payoff of the members of coalition $S' = \{1, 2, 3\}$. As $v(S) = v(S') = 1$, this LME is not separating.

The next theorem characterizes separating limit market equilibria.

Theorem 3. *Let x^1 be a SLME, then,*

(i) all players of the same type cooperate in MCs of homogeneous types. The SLME induces, therefore, a partition of player types according to the coalition type they cooperate in;

(ii) all players of type $t = 1, \dots, T$, who cooperate in coalitions of type $\mathbf{n}$, receive the payoff,

$$x_t^1 = v(\mathbf{n}) \frac{n_t \alpha_t / N_t}{\sum_{s=1}^{T} (n_s^2 \alpha_s / N_s)}. \tag{5}$$

5. Dynamic Labor Market

A natural application of our framework is a labor market where workers (with various skills) bargain over a remuneration scheme, taking into account firm's value and their own (endogenous) outside options. This allows us to compare the theoretical predictions of our model to those derived in the neoclassical competitive setting and by the search theoretic literature. We start off by discussing a labor market with homogeneous workers for which we derive the equilibrium wage and the firm size. Subsequently, we generalize our results to a heterogeneous input market.

5.1. Homogeneous Labor Market

At each date, the labor market consists of one entrepreneur, i, and N homogenous workers where a generic worker is indexed by w. We will denote the set of workers by $\mathcal{N}_w$, hence, the set of all players $\mathcal{N} = \mathcal{N}_w \cup \{i\}$. Each worker has the same bargaining power denoted by α_w. The productivity of a coalition with $n \leq N$ workers and the entrepreneur (a productive coalition) is given by the production function $F(n) : \mathbb{N} \mapsto \mathbb{R}_+$, i.e., $v(S) = F(n)$ for all $S \subseteq \mathcal{N}$ such that $i \in S$ and $|S \cap \mathcal{N}_w| = n$. In particular, production is impossible without workers, $F(0) = 0$, or without the entrepreneur, $v(S) = 0$ for all $S \subseteq \mathcal{N}_w$.

We can compute the ME payoffs for the entrepreneur and a representative worker, x_i^δ and x_w^δ, respectively, from (1) when only coalitions with n workers are matched. For symmetric matching probabilities, system (1) is reduced to two equations,

$$x_i = \delta x_i + \frac{\alpha_i}{\alpha(n)} \max(F(n) - \delta x_i - n\delta x_w, 0), \tag{6}$$

$$x_w = \delta x_w + \frac{\binom{N-1}{n-1}}{\binom{N}{n}} \frac{\alpha_w}{\alpha(n)} \max(F(n) - \delta x_i - n\delta x_w, 0),$$

where $\alpha(n) := \alpha_i + n\alpha_w$. It can be shown that the solution (x_i^δ, x_w^δ) to the latter system satisfies,

$$\frac{x_i^\delta}{x_w^\delta} = \frac{\alpha_i N}{\alpha_w n}, \tag{7}$$

Notably, expression (7) does not depend on the discount factor δ. Instead, it depends in an intuitive way on the relative bargaining power between the entrepreneur and the worker and it is an decreasing function of the relative labor market tightness, n/N where n is the number of vacancies in the market and N measures total labor supply. In particular, when the market is tight (n/N is low) the entrepreneur extracts a higher share of the value of the firm and the opposite holds, when there is relatively more vacancies (relatively larger firm size) than the number of workers looking for a job.

From the solution (x_i^δ, x_w^δ) to (6), we can calculate the limit payoffs,

$$\lim_{\delta \to 1} x_i^\delta = x_i^1 = \frac{\alpha_i F(n) N}{\alpha_i N + \alpha_w n^2}, \tag{8}$$

$$\lim_{\delta \to 1} x_w^\delta = x_w^1 = \frac{\alpha_w F(n) n}{\alpha_i N + \alpha_w n^2},$$

$$x_i^1 + n x_w^1 = F(n).$$

which do not depend on the matching probabilities and are a special case of the limit payoffs specified in item (ii) of Theorem 3. Moreover,

$$\partial x_w^1 / \partial N < 0, \quad \partial x_i^1 / \partial N > 0.$$

The above derivatives are similar to the findings in search models, e.g. Shapiro and Stiglitz (1984): the tighter the labor market (i.e. the higher the labor supply N for a given number of vacancies

n) the lower the worker's wage (x_w^1) is. The reverse is expected to hold with respect to the entrepreneur's payoff.

So far, our assumption that only coalitions with n workers are matched led to the equilibrium payoffs (8). In general, however, all productive coalitions may have a (random) opportunity to cooperate. If we focus on a SLME, a unique firm size and unique limit payoffs (8) emerge endogenously. Recall that the separating property rules out a specific relationship between equilibrium payoffs and coalitional productiveness. Then, item (i) of Theorem 3 implies that only coalitions of a specific size, say with n workers, will be active in a SLME x^1 and that each player type will earn the limit payoff 5. Furthermore, we can re-write item (iii) of Theorem 2 as

$$F(n) - F(n-1) \geq x_w^1 \geq F(n+1) - F(n). \tag{9}$$

Essentially, the above expressions establishes the parity between the separating limit payoff and the neoclassical wage. It entails that the outcome of our multilateral bargaining procedure results in each worker being paid approximately her marginal contribution to the firm value. Given the assumption of homogenous labor, we can interpret the marginal contribution as the marginal product of labor.

To facilitate our discussion of how the equilibrium firm size is related to the market fundamentals, i.e., technology and input abundance, we will assume that the firm is endowed with a Cobb-Douglas production function, $F(n) = An^{\gamma}$, and that the labor input in this function is a continuous variable. In order to simplify the expressions, we focus on a symmetric situation, in which the bargaining power of the entrepreneur is the same as the collective bargaining power of the workers, $\alpha_i = \alpha_w n$.[11] Then the limit equilibrium payoffs (8) simplify to,

$$x_i^1 = \frac{F(n)N}{N+n}, \quad x_w^1 = \frac{F(n)}{N+n}. \tag{10}$$

Recall that (9) indicates that the wage should approximate the product of the marginal worker

[11]Without this assumption, we can establish qualitatively the same results but we will have to deal with more complicated expressions that depend on bargaining powers.

$F'(n)$. Hence, the limit equilibrium firm size must satisfy,[12]

$$x_w^1 = \frac{F(n)}{N+n} = F'(n) = \frac{\gamma}{n} F(n) \Rightarrow n = \frac{N\gamma}{1-\gamma} \quad for \quad \gamma < \frac{1}{2}. \tag{11}$$

Notably, we obtain a full characterization of the SLME - the unique firm size, wages, and profit - without any reference to an exogenous reservation wage (the outside option of the worker). Notice that the equilibrium firm size is larger the greater the total labor supply N. Furthermore, an increase in γ - i.e., a more productive technology - leads to a larger equilibrium firm size n, lower unemployment rate $(N-n)/N$ and a higher equilibrium profit and wage,

$$x_i^1 = N x_w^1, \quad x_w^1 = \frac{F(n)}{N+n} = \frac{A}{N} (N\gamma)^{\gamma} (1-\gamma)^{1-\gamma},$$

as the expression on the r.h.s. of the last equality increases in γ when $n = N\gamma/(1-\gamma) > 1$.

So far in our framework, we have discussed labor market equilibria in the absence of frictions, i.e. $\delta \to 1$. In markets with frictions, we can further replicate the stylized fact that players of the same type may obtain a different expected payoff. This is driven by the matching environment and the fact that players of the same type may reach an agreement in coalitions of distinct types in equilibrium.[13]

5.2. *Labor Market with Heterogeneous Inputs*

This section generalizes the results derived for homogeneous labor markets. At each date, the labor market consists of N_t workers of type $t = 1, \ldots, T$ and one entrepreneur i (player type $T+1$). The production function takes now the form $F(n_1, ..., n_T) : \mathbb{N}^T \mapsto \mathbb{R}_+$, where n_t refers to the number of t-type workers employed by the firm. Alternatively, we can interpret F as the output

[12]Clearly, $\gamma > 1/2$ implies $n > N$. In this case, it can be shown that $n = N$ in the SLME.

[13]As an illustration, consider the following example. Let the set of workers be $\mathcal{N}_w = 1, 2$ and the entrepreneur i. Let the coalition surplus be $v(\{i,1,2\}) = 2$, $v(\{i,1\}) = v(i,2) = 1$, and $v(S) = 0$ for any other $S \subseteq \{i,2,3\}$. Let $\pi_{\{i,1,2\}} = 1/5$, $\pi_{\{i,1\}} = 3/5$, and $\pi_{\{i,2\}} = 1/5$; and $\delta = 0.9$. Then all productive coalitions agree in ME and $x_i = 0.5693$, $x_1 = 0.5052$ and $x_2 = 0.4705$.

of a hierarchy with T levels and n_t employees at each level t. In the unique SLME, only coalitions composed of the entrepreneur and a particular profile $\mathbf{n} = (n_1, ..., n_T)$ of workers are active and players of the same type earn the limit payoffs (5) that exhaust the product. Furthermore, part (ii) of Theorem 2 takes the form,

$$F(\mathbf{n}) - F(\mathbf{n} - \mathbf{e}_t) \geq x_t^1 \geq F(\mathbf{n} + \mathbf{e}_t) - F(\mathbf{n}), \quad \text{for all } t = 1, \ldots, T,$$

which generalizes our findings about the relationship between the SLME payoffs and the neoclassical wage. The inequalities imply that for each type, a worker of that type receives (approximately) her marginal contribution to the firm value. In addition, the SLME payoffs defined in (5) imply that the relative limit wages of different types of workers (different levels in a hierarchy) satisfy,

$$\frac{x_t^1}{x_s^1} = \frac{n_t \alpha_t N_s}{n_s \alpha_s N_t}, \quad s, t = 1, \ldots, T. \tag{12}$$

The latter relationship reveals that the relative scarcity of each type in the market influences the relative wages in the expected way: a higher supply N_t of type t workers depresses their relative individual wages ceteris paribus. On the other hand, the total number n_t of workers of type t, employed in an equilibrium firm, has a positive impact on their relative wages. This is not surprising, when we observe that n_t represents the demand for this type by the equilibrium firm. Therefore, (12) combines in a clear-cut manner the supply and demand forces in the market, adjusted by the type-specific bargaining power.

Next, we consider a Cobb-Douglas production function with T worker types and treat each argument of F as a continuous variable,

$$F(\mathbf{n}) = A n_1^{\gamma_1} \ldots n_T^{\gamma_T}, \quad \text{with } \mathbf{n} = (n_1, \ldots, n_T), \quad \gamma_t > 0, \quad \textstyle\sum_{t=1}^{T} \gamma_t < 1.$$

It will turn out convenient to treat the entrepreneur as a player of type $T + 1$,

$$N_{T+1} = n_{T+1} = 1, \quad \gamma_{T+1} := 1 - \textstyle\sum_{t=1}^{T} \gamma_t.$$

For the sake of simplicity, we assume again equal bargaining power for each group in the equilibrium firm,

$$n_s \alpha_s = n_t \alpha_t, \quad \text{for all } t, s = 1, ..., T + 1. \tag{13}$$

Similar to our analysis of the equilibrium firm size in the homogeneous labor market, we can derive the unique equilibrium firm composition $\mathbf{n}$ by approximating the limit SLME payoff x_t^1 by the marginal contribution of this type,

$$x_t^1 = \frac{\partial F(\mathbf{n})}{\partial n_t} = \frac{\gamma_t}{n_t} F(\mathbf{n}) \text{ for all } t = 1, \ldots, T. \tag{14}$$

As the SLME payoffs (5) exhaust the surplus $F(\mathbf{n})$,

$$F(\mathbf{n}) = x_{T+1}^1 + \textstyle\sum_{t=1}^{T} n_t x_t^1 = x_{T+1}^1 + \sum_{t=1}^{T} \gamma_t F(\mathbf{n}) \Rightarrow x_{T+1}^1 = \gamma_{T+1} F(\mathbf{n}).$$

Using (14), the limit payoffs (5), and the equal bargaining power assumption, (13), we can derive the equilibrium quantity of factor inputs and equilibrium factor payoffs,

$$n_t = \frac{\gamma_t}{\gamma_{T+1}} N_t, \quad x_t^1 = \frac{A}{N_t} \prod\nolimits_{s=1}^{T+1} (\gamma_s N_s)^{\gamma_s}, \quad \text{for } t = 1, \ldots, T+1.$$

Here we assume an interior solution $1 < n_t < N_t$ (or, equivalently, $\gamma_t < \gamma_{T+1}$) for each type $t = 1, \ldots, T$. In an interior SLME, an increase in productivity of type s leads to higher employment levels and higher payoffs for all types of workers and the entrepreneur,

$$\frac{\partial n_t}{\partial \gamma_s} > 0 \quad \text{and} \quad \frac{\partial x_t^1}{\partial \gamma_s} > 0 \quad \text{and} \quad \frac{\partial x_i^1}{\partial \gamma_s} > 0 \quad \text{for all } t, s = 1, \ldots, T.$$

On the other hand, increasing the supply of type s decreases the payoff of this type, while it increases the demand for it and the payoffs to all other players,

$$\frac{\partial x_s^1}{\partial N_s} < 0, \quad \frac{\partial n_s}{\partial N_s} > 0, \quad \frac{\partial x_t^1}{\partial N_s} > 0, \quad \frac{\partial x_i^1}{\partial N_s} > 0, \quad t, s = 1, ..., T, t \neq s.$$

Notice that in the SLMEa discussed above not all coalitions which are matched reach an agreement. Thus some workers choose to remain unemployed in anticipation of higher wages and some firms choose to keep vacancies unfilled in the anticipation of higher profits. This insight complies with the stylized facts found in the search-theoretic literature, cf. Rogerson et al. (2005). We derive these results, however, in an environment where market frictions, as captured by the matching probabilities, do not matter for the equilibrium behavior. Moreover, in our framework positive equilibrium level of unemployment co-exists with the neoclassical competitive equilibrium rule that is rewarding all inputs their marginal product.

6. Concluding remarks and extensions

We have developed a price-setting mechanism that makes explicit the role of strategic behavior in the context of dynamic multilateral markets. We have shown the existence and payoff uniqueness of SSPE in any multilateral market game in which players exhibit at least some degree of impatience. Furthermore, we have studied in more detail the implications of our theoretical framework for the equilibrium price in the labor market. Unlike, other price-setting mechanisms based on multilateral bargaining applied to the labor market, we find that our procedure results in equilibrium prices which equal the respective marginal product of the factors of production. In this respect, we see our model as providing a strategy-based microeconomic alternative to the Walrasian auctioneer's procedure to find the competitive prices. Moreover, our model allows for an endogenous determination of the composition of a firm with heterogeneous inputs without any reference to the outside options of players. In this setting, we find support for a number of stylized facts discussed in the search-theoretic models, such as the existence of voluntary unemployment and unfilled vacancies, the dependence of wages on the relative scarcity of skill type, and that workers of homogeneous skill types may obtain different equilibrium payoffs.

Further extensions to the current study of the labor market are possible. Similar to Stole and Zwiebel (1996a) one can address the question of organization design in the context of multilateral coalitions and investigate how the interplay of factors' productivity, factors' abundance, and bargaining power shape up the equilibrium outcome. In such setting the hierarchical depth and width of the firm may be derived endogenously in market equilibrium as a function of the market fundamentals.

We also want to stress the straightforward applicability of our results to markets that exhibit multilateral structure of interactions, such as the housing market, credit-card market or retailing. For example, an application of (5) to a market game with T types of players, where only coalitions composed of one player of each type are productive, results in unique SLME payoffs.[14] Similar

[14] In this SLME $x_t^1 = \frac{\alpha_t/N_t}{\sum_s(\alpha_s/N_s)}, \quad \frac{x_t^1}{x_s^1} = \frac{\alpha_t N_s}{\alpha_s N_t} \quad$ for $t, s = 1, \ldots, T$.

intuition to the analysis of two-sided markets can be derived here, too. A player's payoff is increasing with her bargaining power and with the relative weighted scarcity of players of her own type with weights equal to the bargaining power of each type. Moreover, the equilibrium payoff of each player is positively related to the number of players of the opposite type and negatively related the number of players of her own type.

7. Appendix

PROOF. *Theorem 1*:

The proof boils down to showing that the mapping $f : R^N \to R^N$ on the r.h.s. of (1),

$$f_i(x) := \delta x_i + \sum_{S\in\mathcal{S}_i} \pi_S \frac{\alpha_i}{\alpha(S)} \mathbf{I}_{\mathbf{v(S)}\geq\delta\mathbf{x(S)}}(v(S) - \delta x(S)), \quad \forall i \in \mathcal{N}, \tag{15}$$

where $\mathbf{I_c} = 1$ if c is true and $\mathbf{I_c} = 0$ otherwise, is a contraction. We can write f at point $x \in \mathbb{R}^N$ in matrix form,

$$f(x) = f^x(x) := (\delta I - \delta D_\alpha \Pi^x)x + b^x, \quad I, D_\alpha, \Pi^x \in R_+^{N\times N}, b^x \in R_+^N,$$

$$I \text{ is } N \times N \text{ identity matrix}, \quad (D_\alpha)_{ij} = \mathbf{I_{i=j}}\alpha_i, \quad b_i^x = \alpha_i \sum\nolimits_{S\in\mathcal{S}_i} \gamma_S v(S) \mathbf{I}_{\mathbf{v(S)}\geq\delta\mathbf{x(S)}},$$

$$\Pi_{ij}^x = \sum\nolimits_{S\in\mathcal{S}_i\cap\mathcal{S}_j} \gamma_S \mathbf{I}_{\mathbf{v(S)}\geq\delta\mathbf{x(S)}} = \sum\nolimits_{S\subseteq\mathcal{N}} \gamma_S \mathbf{I_{i\in S}} \mathbf{I_{j\in S}} \mathbf{I}_{\mathbf{v(S)}\geq\delta\mathbf{x(S)}},$$

$$\gamma_S = \pi_S/\alpha(S) \geq 0, \quad \forall i, j \in \mathcal{N}.$$

Hence, D_α is a diagonal matrix and Π^x is a non-negative symmetric matrix (note that $D_\alpha\Pi^x$ is not necessarily symmetric). We observe that the maximum eigenvalue of $D_\alpha\Pi^x$ is bounded from above by one, the highest possible sum of each column j of $D_\alpha\Pi^x$,

$$\forall j \in \mathcal{N}, \quad \sum\nolimits_{i=1}^N (D_\alpha\Pi^x)_{ij} = \sum\nolimits_{i=1}^N \alpha_i \sum\nolimits_{S\in\mathcal{S}_i\cap\mathcal{S}_j} \gamma_S \mathbf{I}_{\mathbf{v(S)}\geq\delta\mathbf{x(S)}}$$

$$\leq \sum\nolimits_{i=1}^N \alpha_i \sum\nolimits_{S\in\mathcal{S}_i\cap\mathcal{S}_j} \gamma_S = \sum\nolimits_{S\in\mathcal{S}_j} \pi_S \leq 1$$

$$\Rightarrow \lambda_{\max}(D_\alpha\Pi^x) \leq ||D_\alpha\Pi^x||_1 \leq 1.$$

In order to show the lower bound, $\lambda_{\min}(D_\alpha \Pi^x) \geq 0$, we note first that Π^x is positive semidefinite,

$$\forall z \in R^N, \quad z^T \Pi^x z = \sum_{S \subseteq \mathcal{N}} \left(\gamma_S \mathbf{I}_{\mathbf{v(S)} \geq \delta \mathbf{x(S)}} \sum_i \sum_j z_i z_j \mathbf{I}_{\mathbf{i} \in \mathbf{S}} \mathbf{I}_{\mathbf{j} \in \mathbf{S}} \right)$$
$$= \sum_{S \subseteq \mathcal{N}} \gamma_S \mathbf{I}_{\mathbf{v(S)} \geq \delta \mathbf{x(S)}} \left(\sum_i z_i \mathbf{I}_{\mathbf{i} \in \mathbf{S}} \right)^2 \geq 0,$$

which implies that all eigenvalues of Π^x are nonnegative. Furthermore, $D_\alpha \Pi^x$ and $D_\alpha^{1/2} \Pi^x D_\alpha^{1/2}$ have the same set of eigenvalues because

$$(D_\alpha^{1/2} \Pi^x D_\alpha^{1/2}) z = \lambda z \Leftrightarrow (D_\alpha \Pi^x) \widetilde{z} = \lambda \widetilde{z}, \quad \text{where} \quad \widetilde{z} := D_\alpha^{1/2} z.$$

As Π^x is symmetric, it can be diagonalized by the orthogonal matrix P,

$$D_\alpha^{1/2} \Pi^x D_\alpha^{1/2} = D_\alpha^{1/2} (P \Lambda P^T) D_\alpha^{1/2} = (D_\alpha^{1/2} P) \Lambda (D_\alpha^{1/2} P)^T.$$

By Sylvester's law of inertia, the number of negative eigenvalues is the same for Λ and for $(D_\alpha^{1/2} P)$ Λ $(D_\alpha^{1/2} P)^T$. As the former diagonal matrix has only nonnegative entries (eigenvalues of Π^x), the same must hold for the latter. We showed, therefore, the bounds on the eigenvalues of $D_\alpha \Pi^x$ and, incidentally, on the corresponding eigenvalues of $\delta I - \delta D_\alpha \Pi^x$,

$$0 \leq \lambda_i(D_\alpha \Pi^x) \leq 1 \Rightarrow$$
$$0 \leq \lambda_i(\delta I - \delta D_\alpha \Pi^x) = \delta(1 - \lambda_i(D_\alpha \Pi^x)) \leq \delta.$$

We conclude that f^x is a contraction,

$$\forall x, v, y \in R^N, \quad ||f^x(v) - f^x(y)|| \leq \delta ||v - y||.$$

Now, in order to show that f contracts $v, y \in R^N$, we choose on the segment $(v, y) \subset R^N$ a sequence $\{v = z_0, z_1, ..., z_n, z_{n+1} = y\}$ of all points with increasing distances from v such that any $\epsilon-$neighborhood of z_k, $k = 1, .., n$, with radius $\epsilon > 0$ contains points $z_k^1(\epsilon)$ and $z_k^2(\epsilon)$ in (v, y) with $f^{z_k^1(\epsilon)} \neq f^{z_k^2(\epsilon)}$. Note that this sequence is finite as (v, y) can cross only a finite number of (intersections of) hyperplanes of the form $v(S) = \delta z(S)$. By construction, any point $z_{k,k+1} \in$

(z_k, z_{k+1}), $k = 0, .., n$, induces the same linear contraction $f^{z_{k,k+1}}$. Then, the continuity of f implies that,

$$f(z_k) = f^{z_{k,k+1}}(z_k) = f^{z_{k-1,k}}(z_k),$$

where only the first (second) equality holds for $k = 0$ ($k = n + 1$). In particular, for the points $\{z_0, z_1, z_2\}$, we obtain,

$$||f(z_0) - f(z_1)|| = ||f^{z_{0,1}}(z_0) - f^{z_{0,1}}(z_1)|| \leq \delta||z_0 - z_1||,$$
$$||f(z_1) - f(z_2)|| = ||f^{z_{1,2}}(z_1) - f^{z_{1,2}}(z_2)|| \leq \delta||z_1 - z_2|.$$

By the triangular property and the above inequalities,

$$||f(z_0) - f(z_2)|| \leq ||f(z_0) - f(z_1)|| + ||f(z_1) - f(z_2)||$$
$$\leq \delta||z_0 - z_1|| + \delta||z_1 - z_2|| = \delta||z_0 - z_2||,$$

where the last equality follows because $\{z_0, z_1, z_2\}$ lie on the same line. Hence, f contracts $\{z_0, z_2\}$ and we can iterate this argument for $\{z_0, z_3\}$,..., $\{z_0, z_{n+1}\} = \{v, y\}$. ■

PROOF. *Theorem 2*:

(i) First, we will show that all players of the same type receive the same payoff in a LME. Consider a LME x^1 and, for the sake of contradiction, let $x_i^1 > x_j^1$ for two distinct players $i, j \in \mathcal{N}$ with $i \neq j$ who are of the same type, i.e., $T(\{i\}) = T(\{j\})$. Then, for all $S \subseteq \mathcal{N} \setminus \{i, j\}$ and some δ close enough to 1, it holds that

$$\pi_{S\cup\{i\}} \frac{\alpha_i}{\alpha(S \cup \{i\})} \Big(v(S \cup \{i\}) - \delta x^\delta(S \cup \{i\})\Big) \leq$$
$$\pi_{S\cup\{j\}} \frac{\alpha_j}{\alpha(S \cup \{j\})} \Big(v(S \cup \{j\}) - \delta x^\delta(S \cup \{j\})\Big)$$

as $T(\{i\}) = T(\{j\})$ implies that $\alpha_i = \alpha_j$ and for all $S \subseteq \mathcal{N} \setminus \{i, j\}$, $\pi_{S\cup\{i\}} = \pi_{S\cup\{j\}}$, $\alpha(S \cup \{i\}) = \alpha(S \cup \{j\})$, and $v(S \cup \{i\}) = v(S \cup \{j\})$. The above inequality holds as an equality if and only if $\pi_{S\cup\{i\}} = \pi_{S\cup\{j\}} = 0$.

In addition, notice that for all $S \subseteq \mathcal{N}$ such that $i \in S$ and $j \in S$

$$\pi_S \frac{\alpha_i}{\alpha(S)} \left(v(S) - \delta x^\delta(S) \right) = \pi_S \frac{\alpha_j}{\alpha(S)} \left(v(S) - \delta x^\delta(S) \right).$$

Therefore for a δ close enough to 1, we can use (1) and the above analysis to write

$$\begin{aligned}
(1-\delta)x_i^\delta &= \sum_{S \in \mathcal{S}_i} \pi_S \frac{\alpha_i}{\alpha(S)} \max\{v(S) - \delta x^\delta(S), 0\} \\
&\leq \sum_{S \in \mathcal{S}_j} \pi_S \frac{\alpha_j}{\alpha(S)} \max\{v(S) - \delta x^\delta(S), 0\} \\
&= (1-\delta)x_j^\delta.
\end{aligned}$$

This leads to a contradiction as by assumption $x_i^\delta > x_j^\delta$ for $\delta \to 1$.

Thus, we can index by x_t^1 the LME payoff for all players of type t and all types $t = 1, \ldots, T$.

(ii) Next, we will show that for any active coalition of type $\mathbf{n} = (n_1, \ldots, n_T)$ in a LME x_1, it holds that $\sum_t n_t x_t^1 = v(\mathbf{n})$.

Recall that for any active coalition $S \subseteq \mathcal{N}$ in a ME x^δ, it holds that $v(S) - \delta x^\delta \geq 0$. We proceed by establishing a contradiction. Thus we assume that there exists a coalition $S \subseteq \mathcal{N}$ such that $\lim_{\delta \to 1}(v(S) - \delta x^\delta(S)) > 0$. Then taking the limit of (1) for a player $i \in S$ and re-arranging we obtain

$$\underbrace{\lim_{\delta \to 1}(1-\delta)x^\delta}_{=0} = \underbrace{\lim_{\delta \to 1} \sum_{S \in \mathcal{S}_i} \frac{\alpha_i}{\alpha(S)} \max\{v(S) - \delta x^\delta, 0\}}_{>0}.$$

As this leads to a contradiction, it must be that for all active coalitions S in the LME x^1, $\lim_{\delta \to 1}(v(S) - \delta x^\delta(S)) = 0$. Thus, using the fact that all players of the same type receive equal limit payoffs, we have established that for all active coalitions of type $\mathbf{n} = (n_1, \ldots, n_T)$ in a LME x_1, it holds that $\sum_t n_t x_t^1 = v(\mathbf{n})$.

(iii) Consider a player of type t who cooperates in a LME x^1 in a coalition S with $T(S) = \mathbf{n}$. Then, by (i), we can write

$$x^1(S) = v(S) = v(\mathbf{n}) = \textstyle\sum_{t=1}^T n_t x_t^1, \quad x^1(S') \geq v(S'), \quad \forall S' \subseteq \mathcal{N}.$$

The latter inequality must hold, in particular, for

$$S' = S^- : T(S^-) = \mathbf{n} - \mathbf{e}_t \quad \& \quad S' = S^+ : T(S^+) = \mathbf{n} + \mathbf{e}_t.$$

where $v(\mathbf{n} - \mathbf{e}_t)$ and $v(\mathbf{n} + \mathbf{e}_t)$ are well-defined values. Then,

$$x^1(S^-) = \textstyle\sum_{s \neq t} n_s x^1_s + (n_t - 1)x_t = v(\mathbf{n}) - x_t \geq v(\mathbf{n} - \mathbf{e}_t) = v(S^-),$$
$$x^1(S^+) = \textstyle\sum_{s \neq t} n_s x^1_s + (n_t + 1)x_t = v(\mathbf{n}) + x_t \geq v(\mathbf{n} + \mathbf{e}_t) = v(S^+),$$

which yields the claim. ■

Before we discuss the proof of Theorem 3, we establish the following supplementary results.

Lemma 1. *In a SLME x^1,*

$$\forall S, S' : T(S) \neq T(S'), S \cap S' \neq \varnothing, \quad x^1(S) = v(S) \Rightarrow x^1(S') > v(S').$$

PROOF. First, notice that the case $x^1(S) = v(S) \Rightarrow x^1(S') < v(S')$ is ruled out by Theorem 2 item (i).

Next, consider the case $x^1(S) = v(S) \Rightarrow x^1(S') = v(S')$. Then one of the following three cases must hold:

$$v(S) > 0 \text{ and } v(S') > 0 \Rightarrow \frac{x^1(S)}{v(S)} = \frac{x^1(S')}{v(S')} \Rightarrow x^1(S)v(S') = x^1(S')v(S),$$
$$v(S) \geq v(S') = 0 \Rightarrow x^1(S) \geq x^1(S') = 0 \Rightarrow x^1(S)v(S') = x^1(S')v(S) = 0,$$
$$v(S') \geq v(S) = 0 \Rightarrow x^1(S') \geq x^1(S) = 0 \Rightarrow x^1(S)v(S') = x^1(S')v(S) = 0.$$

As shown all of the above three cases contradict the separating property of x^1, which completes the proof. ■

Lemma 2. *If the ME x^δ implies that all players of types t and s with $s, t \in \{1, \ldots, T\}$ cooperate only in coalitions of type $\mathbf{n} = (n_1, ..., n_T)$, then,*

$$n_s \alpha_s x^\delta(\mathcal{N}_t) = n_t \alpha_t x^\delta(\mathcal{N}_s).$$

PROOF. By summing up (1) over all t-type players for some $t \in \{1, \ldots, T\}$, one obtains the total payoff of the players of this type,

$$\begin{aligned} x^{\delta}(\mathcal{N}_t) &= \delta x^{\delta}(\mathcal{N}_t) + n_t \alpha_t \sum\nolimits_{S:T(S)=\mathbf{n}} \pi_S \frac{v(S) - \delta x^{\delta}(S)}{\alpha(S)} \\ &= \frac{n_t \alpha_t}{1-\delta} \sum\nolimits_{S:T(S)=\mathbf{n}} \pi_S \frac{v(S) - \delta x^{\delta}(S)}{\alpha(S)} =: \frac{n_t \alpha_t}{1-\delta} \Delta^{\delta}(\mathbf{n}). \end{aligned}$$

Note that we use the fact that the bargaining power α_t is the same for all players of type t. By the same argument, the total payoff to the players of type $s \in \{1, \ldots, T\}$ is $x^{\delta}(\mathcal{N}_s) = n_s \alpha_s \Delta^{\delta}(\mathbf{n})/(1-\delta)$ and the claim follows. ■

PROOF. *Theorem 3*:

(i) To show that in all SLMEa, all players of the same type cooperate in MCs of homogeneous types, we will proceed by establishing a contradiction. Consider a SLME x^1. For the sake of contradiction assume that in this SLME players i and j of type $t \in \{1, \ldots, T\}$ cooperate in coalitions S and S', respectively, with $T(S) = (n_1, ..., n_T) \neq (n'_1, ..., n'_T) = T(S')$. Let the coalition S'' be the same as S' except for player i who replaces player j, i.e. $S'' = S' \cup \{i\} \setminus \{j\}$. Hence, $T(S) \neq T(S') = T(S'')$ and $\{i\} \in S \cap S''$. As S and S' are active, Theorem 2, items (i) and (ii), and the fact that $T(S') = T(S'')$ imply,

$$v(S) = x^1(S), \quad x^1(S') = v(S') = v(S'') = x^1(S'').$$

However, this contradicts, by Lemma 1, the separating property of x^1 as $S \cap S'' \neq \varnothing$ and $T(S) \neq T(S'')$.

(ii) Consider a SLME x^1. We have established that each player of type $t \in \{1, \ldots, T\}$ cooperates in MCs of homogeneous types. By Lemma 2, the total payoff for all players of type t and all players of type $s \in \{1, \ldots, T\}$, that cooperate in MCs of the same type as a type $t-$ player, i.e., $\mathbf{n} = (n_1, ..., n_T)$, satisfy

$$n_s \alpha_s x^{\delta}(\mathcal{N}_t) = n_t \alpha_t x^{\delta}(\mathcal{N}_s).$$

This equality must hold also for the SLME x^1. Thus using Theorem 2 item (ii), we can re-write the last equality as

$$n_s \alpha_s N_t x_t^1 = n_t \alpha_t N_s x_s^1. \tag{16}$$

In particular, $x_t^1 = 0$ implies $x_s^1 = 0$ for any two distinct types that cooperate in a coalition S such that $T(S) = \mathbf{n}$. This is only possible in a SLME if $v(S) = v(\mathbf{n}) = 0$.

Suppose $x_t^1 > 0$ for some type t. Then, by Theorem 2 items (i) and (ii), it follows that

$$v(S) = v(\mathbf{n}) = x^1(S) = \sum\nolimits_{s=1}^{T} n_s x_s^1.$$

Using (16) to substitute for x_s^1,

$$v(\mathbf{n}) = \sum\nolimits_{s=1}^{T} \frac{n_s^2 \alpha_s N_t x_t^1}{n_t \alpha_t N_s}.$$

By re-arranging the above expression, one obtains the SLME payoff of type t-players,

$$x_t^1 = v(\mathbf{n}) \frac{n_t \alpha_t / N_t}{\sum_{s=1}^{T} (n_s^2 \alpha_s / N_s)}. \qquad \blacksquare$$

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