Web Appendix

Appendix 1

Proposition 1: (a) In the optimal plan retirement occurs once and is not reversed.

Given our assumptions the optimal plan for the agent is to begin life working, retire at an age R, and not to work thereafter.

At point R the agent is indifferent between working and retiring and the equality condition of equation (17) holds. Given equation (16) and the fact that $r=\delta$, optimal consumption is constant over the life span. Now consider the function

$$\psi(R)=u'(c)w_1(R)w_2(h(R,z))-v(h(R,z)) \tag{27}$$

Clearly for R to be the optimal retirement and to satisfy equation (17) we require $\psi(R)=0$. We wish to show that only one R can satisfy the condition that $\psi(R)=0$ so the retirement age is unique and retirement is permanent. We do this by showing that $\psi'(R)<0$ whenever $\psi(R)=0$. Thus every point where $\psi(R)=0$ is a transition from working to retirement and re-entry to the labor market once retired is impossible.

We proceed by contradiction. Assume $\psi'(R)\geq 0$. We have

$$\begin{aligned}\psi'(R)&=u'(c(R))\frac{dw_1(R)}{dR}w_2(h(R,z))\\&+u'(c(R))w_1(R)\frac{dw_2(h(R,z))}{dh}\frac{dh(R,z)}{dR}-\frac{dv(h(R,z))}{dh}\frac{dh(R,z)}{dR}\geq 0\end{aligned} \tag{28}$$

Hence at any R with $\psi(R)=0$ we have

$$\begin{aligned}&\frac{d\log(w_1(R))}{dR}+\frac{d\log(w_2(h(R,z)))}{dh}\frac{dh(R,z)}{dR}\\&\geq\frac{d\log(v(h(R,z)))}{dR}\end{aligned} \tag{29}$$

Because $h(R,z)$ is decreasing in R while $w_2(h)$ is increasing in h, the second term on the left hand side of equation (29) is negative. Because the growth of the exogenous component of wages and the disutility of labor with respect to the retirement age is the same as with respect to time, equation (29) implies

$$\frac{\dot{w}_1}{w_1} \geq \frac{\dot{v}}{v} \tag{30}$$

which contradicts the assumption in equation (11).

Hence $\psi'(R) < 0$ at every age R with $\psi(R) = 0$. It follows that ages that satisfy the first-order condition (where $\psi(R) = 0$) are isolated. Now take two adjacent points $R_1 < R_2$ that satisfy $\psi(R) = 0$. By the fact that the derivative of ψ is negative at both points there exists ε small such that $\psi(R_1 + \varepsilon) < 0$ and $\psi(R_2 - \varepsilon) > 0$. Hence by the intermediate value theorem there exists R_3 between R_1 and R_2 with $\psi(R_3) = 0$. This contradicts R_1 and R_2 being adjacent. It follows that there can be at most one age R satisfying $\psi(R) = 0$ and that the worker works up to this age and does not work afterwards. We call R the retirement age.

Now consider the boundary conditions. These imply the worker does not work at T. Now suppose the worker never works. Then consumption is always zero and the other boundary condition implies that the worker prefers to work at time zero, a contradiction.

Therefore the worker starts life working, and retirement occurs once and is permanent.

(b) The retirement age and consumption stream that maximize expected lifetime utility are unique.

Note that in the proof of part (a) we used the fact that in an optimal plan consumption is flat over time. We now, however, compare two optimal plans, one with a longer life expectancy and one with a shorter one. By part (a) each optimal plan has a unique retirement age. From the budget constraint it is clear that a later retirement age will generate higher lifetime consumption. Again we consider the function $\psi(R)$ over different possible retirement ages

$$\psi(R) = u'(c(R))w_1(R)w_2(h(R,z)) - v(h(R,z)) \tag{31}$$

We require $\psi(R) = 0$ at an optimal retirement age. Once again, assuming $\psi'(R) \geq 0$, we have

$$\begin{aligned} \psi'(R) &= u''(c(R))\frac{dc(R)}{dR}w_1(R)w_2(h(R,z)) + u'(c(R))\frac{dw_1(R)}{dR}w_2(h(R,z)) \\ &+u'(c(R))w_1(R)\frac{dw_2(h(R,z))}{dh}\frac{dh(R,z)}{dR} - \frac{dv(h(R,z))}{dh}\frac{dh(R,z)}{dR} \geq 0 \end{aligned} \tag{32}$$

Now, however, we have $\frac{dc(R)}{dR}$ >0 because we are comparing retirement over different optimal paths rather multiple retirement ages within a single optimal path. Using the fact that $\psi(R) = 0$ at each retirement age we now have

$$\left[\frac{-1}{\theta(c(R))}\right]\frac{d\log(c(R))}{dR} + \frac{d\log(w_2(h(R,z)))}{dh}\frac{dh(R,z)}{dR} + \frac{\dot{w}_1}{w_1} \geq \frac{\dot{v}}{v} \tag{33}$$

where $\theta(c(R))$ is the local inter-temporal elasticity of substitution at retirement, which is positive because the utility function is concave. Because $c(R)$ is increasing and $w_2(R,z)$ is decreasing in R, the first two terms on the left hand side of equation (33) are negative. Hence equation (30) must again hold, which contradicts our assumptions. It follows that in this case we also have $\psi'(R) < 0$ at each optimal retirement age. By a similar argument to that in part (a) we have that $\psi(R) = 0$ only once and that there is only one optimal plan for the agent.

Appendix 2

Proportionality Condition:

To investigate this we look at the empirical survival schedules for the United States. Let us assume that there is no health effect on wages; the health effect will increase wage income when life expectancy rises making the proportionality condition easier to satisfy. We assume real wages grow at a constant rate σ. Then the budget constraint gives the initial ratio of consumption to wages at the original survival schedule

$$\frac{c_0^*}{w_0} = \frac{\int_0^{R_0^*} e^{(\sigma-r)t} s(t, z_0)\, dt}{\int_0^{\infty} e^{-rt} s(t, z_0)\, dt} \tag{34}$$

This implies that the proportionality condition on the initial consumption-wage ratio is:

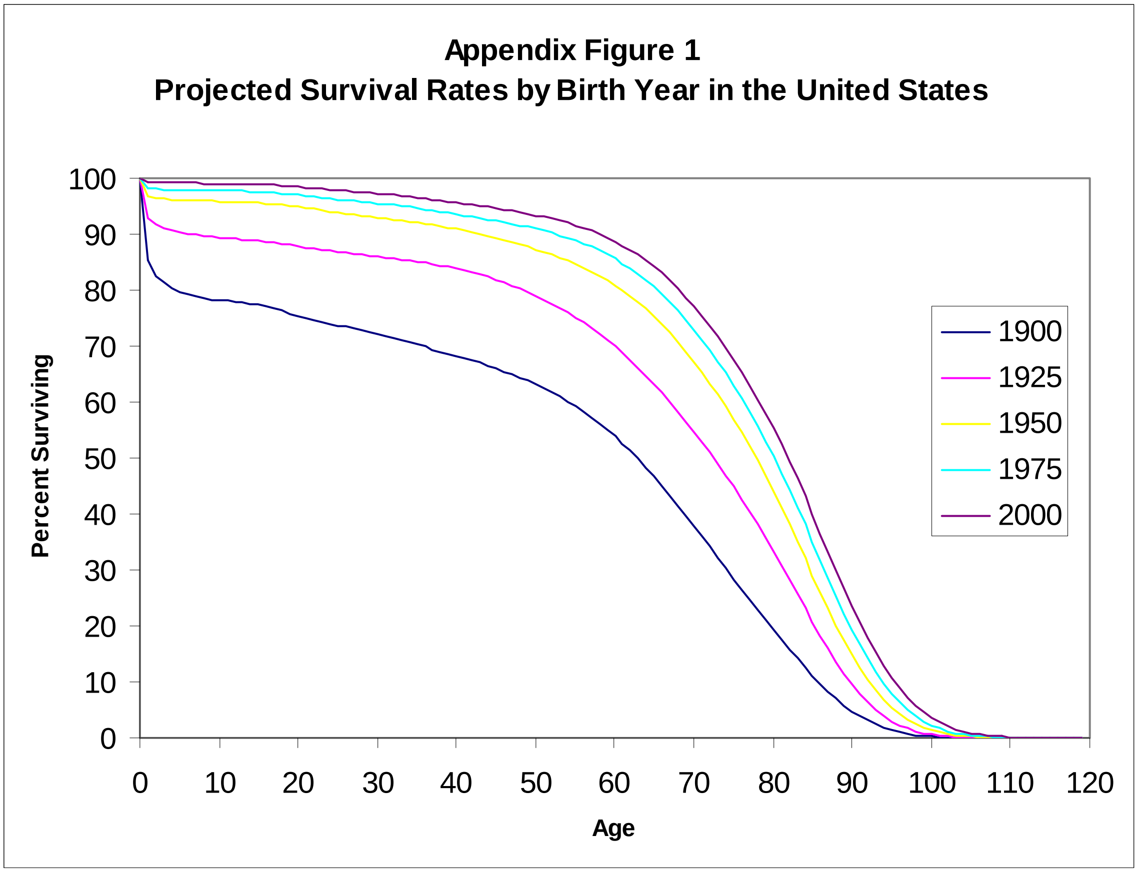
$$\frac{\int_0^{R_0^* z_1 / z_0} e^{(\sigma-r)t} s(t, z_1)\, dt}{\int_0^{\infty} e^{-rt} s(t, z_1)\, dt} \geq \frac{\int_0^{R_0^*} e^{(\sigma-r)t} s(t, z_0)\, dt}{\int_0^{\infty} e^{-rt} s(t, z_0)\, dt} \tag{35}$$

This can be easily checked given two survival schedules provided that we have data for the real interest rate, the rate of real wage growth, and the initial retirement age. In its fiscal forecasts for Social Security in the US, the Congressional Budget Office (2004) uses historical averages for the real interest rate of 3.30% per annum and of the rate of real wage growth of 1.27% per annum. We use the same figures. Gendell (2001) shows that between 1950 and 2000 the mean retirement age for men fell from just under 69 years to just under 63 years of age. Costa (1998a) shows that labor force participation rates for men over 65 before 1900 were around 80% but had fallen to around 50% by 1950, suggesting a high mean age of retirement before 1900. We take a range of possible figures for the retirement age in 1900 (80, 70, and 60 years of age) to allow for variations across individuals.

We use data on survival rates for cohorts born in 1900, 1925, 1950, 1975, and 2000 based on mortality data and projections from the Berkeley Mortality Database.[11] Appendix Figure 1 shows the survival curves for each of these cohorts. The life expectancy of men in these cohorts rises from 51.5 years in 1990 to 78.2 by the year 2000.[12]

[11] The database is available online at http://www.demog.berkeley.edu/~bmd/

[12] These cohort life expectancies are larger than the corresponding period life expectancies because the cohort figures take into account falling mortality rates over time as the cohort ages, while the more common period life expectancies are based on current cross sectional mortality rates.



Source: Berkeley Mortality Database

For each fixed retirement age we ask if a proportional increase in the age of retirement (proportional to life expectancy) would allow the same consumption stream as before if given a later survival schedule. Appendix Table 1 shows the results from such an exercise starting in 1900. We assume that adult life starts at age 16. The first row in the table gives the years we use for comparison. The second row shows life expectancy at age 16 in each year. The third row gives the ratio of life expectancy at age 16 in each year relative to the figure for 1900. This ratio is how much the retirement age has to be increased if the ratio of working life to adult life is to be kept constant.

Appendix Table 1

Testing the Proportionality Condition Relative to the 1900 Survival Schedule

	Retirement Age in 1900	Year of Survival Schedule				
Year		1900	1925	1950	1975	2000
Life Expectancy at Age 16		50.7	56.4	59.6	61.8	63.5
Life Expectancy Relative to 1900		1.00	1.11	1.18	1.22	1.25
Initial Consumption-Wage Ratio with retirement age proportional to life expectancy	60	1.14	1.19	1.22	1.25	1.26
	70	1.23	1.27	1.30	1.31	1.32
	80	1.28	1.31	1.32	1.33	1.34

For each retirement age, the proportionality condition is satisfied if the initial consumption wage ratio under the new survival schedule, allowing for an increase in the retirement age proportional to the increase in life expectancy, is no smaller than that with the 1900 survival schedule.

Source: Berkeley Mortality Database and the authors' calculations.

We now turn to our calculation of the initial consumption-wage ratios. The third column of the fourth row gives the initial consumption to wage ratio (at age 16) for someone born in 1900 who starts working at 16 and retires at age 60. This is calculated using the right-hand side of equation (34) based on the survival schedule for someone born in 1900, with a real interest rate of 3.3% and real wage growth of 1.27% per year. Note that the initial consumption-wage ratio is 1.14 so that the worker (initially) consumes 14% more than his wage income. This is a common feature of optimal consumption when real wages are rising. The higher expected wages later in working life allow high initial consumption. The young worker borrows initially to allow high consumption but consumes less than his wage income as an older worker to repay this borrowing and save for retirement.

The next column of row 4 shows the initial consumption-wage ratio the worker could afford if he had the survival schedule of someone born in 1925, but also increased his retirement age by 11% to compensate for the longer adult life expectancy. With this longer working life and total life span, the initial consumption-wage ratio can rise to 1.19 indicating that the proportionality condition is satisfied. We repeat the analysis comparing the cohort born in 1900 with the cohorts born in 1950, 1975, and 2000 and in each case find the proportionality condition is satisfied. We also find that in pair-wise comparisons (not reported) of earlier versus later cohorts based on the survival schedules for cohorts born in 1925, 1950, 1975, and 2000, the proportionality condition still holds.

Appendix 3

Proposition 2: If changes in the survival schedule satisfy the proportionality condition, the optimal response to an increase in life expectancy requires consumption to stay the same or increase.

Proof. Let R_o^*, c_0^* be the optimal retirement-consumption plan at life expectancy z_0^* and let R_1^*, c_1^* be the optimal retirement-consumption plan at life expectancy z_1^*. We derive a proof by contradiction.

Let us assume $c_1^* < c_0^*$. Let $z_1 = \theta z_0$ where $\theta > 1$. We must have $R_1^* < R_0^* z_1 / z_0 = \rho R_0^*$, giving more leisure, else the optimal plan cannot be as good as the proportional plan (c_0^*, pR_0^*) , which is feasible. At the optimal retirement ages from the first-order condition (17) we have

$$u'(c_0^*) w_1(R_0^*) = \frac{v(h(R_0^*, z_0))}{w_2(h(R_0^*, z_0))}$$

$$u'(c_1^*) w_1(R_1^*) = \frac{v(h(R_1^*, z_1))}{w_2(h(R_1^*, z_1))}$$

(a) We first take the case where $R_1^* \geq R_0^*$. Because $R_1^* < \rho R_0^*$, v is strictly increasing and w_2 is strictly decreasing it follows that

$$\frac{v(h(R_1^*, z_1))}{w_2(h(R_1^*, z_1))} < \frac{v(h(\rho R_0^*, \rho z_0))}{w_2(h(\rho R_0^*, \rho z_0))} = \frac{v(h(R_0^*, z_0))}{w_2(h(R_0^*, z_0))}$$

Hence $u'(c_1^*) w_1(R_1^*) < u'(c_0^*) w_1(R_0^*)$. Further, because $R_1^* \geq R_0^*$ we have $w_1(R_1^*) \geq w_1(R_0^*)$ because w_1 is increasing. Hence $u'(c_1^*) < u'(c_0^*)$ and so $c_1^* > c_0^*$, a contradiction.

(b) Now we examine the case $R_1^* < R_0^*$. Again assume $c_1^* < c_0^*$.

$$u'(c_1^*) w_1(R_0^*) > u'(c_0^*) w_1(R_0^*) = \frac{v(h(R_0^*, z_0))}{w_2(h(R_0^*, z_0))} > \frac{v(h(R_0^*, z_1))}{w_2(h(R_0^*, z_1)}$$

By our optimality conditions (17) this implies that the agent with life expectancy z_1 works at time R_0^*. This is a contradiction with $R_1^* < R_0^*$ and proposition 1 which ensures non-re-entry into the labor market after retirement age is unique. Hence if $R_1^* < R_0^*$. we must have $c_1^* \geq c_0^*$.

Appendix 4

Proposition 3: If changes in the survival schedule satisfy the proportionality condition, $h(\rho t, \rho z) = h(t, z)$, and there is no exogenous wage growth, the optimal response to an increase in life expectancy requires the retirement age to rise proportionately, or less than proportionately, with life expectancy.

Proof: We prove by contradiction. Let R_0^*, c_0^* be the optimal retirement-consumption plan at life expectancy z_0^* and let R_1^*, c_1^* be the optimal retirement-consumption plan at life expectancy z_1^*. Set $\rho = z_1 / z_0$. Assume that $R_1^* > \rho R_0$. Hence $h(R_1, z_1) < h(\rho R_0, \rho z_0) = h(R_0, z_0)$.

At the optimal retirement ages we have from the first-order condition (17)

$$u'(c_0^*) w_1(R_0^*) = \frac{v(h(R_0^*, z_0))}{w_2(h(R_0^*, z_0))}$$

$$u'(c_1^*) w_1(R_1^*) = \frac{v(h(R_1^*, z_1))}{w_2(h(R_1^*, z_1))}$$

Since v is strictly increasing and w_2 is strictly decreasing

$$\frac{v(h(R_1^*, z_1))}{w_2(h(R_1^*, z_1))} > \frac{v(h(R_0^*, z_0))}{w_2(h(R_0^*, z_0))}$$

Since there is no exogenous wage growth (that is $w_1\left(R_0\right) = w_1\left(R_1\right)$) this implies $u'\left(c_1^*\right) > u'\left(c_0^*\right)$ which in turn implies $c_1 < c_0$. But this contradicts proposition 2. Hence $R_1^* \leq \rho R_0$

Appendix 5

Proposition 4: When the wage schedule rises the optimal response of the agent is to

(i) keep retirement unchanged and increase consumption proportionately with wages if the utility function has a local inter-temporal elasticity of substitution of unity.

(ii) increase the age of retirement and increase consumption more than proportionately with wages if the utility function has a local inter-temporal elasticity of substitution greater than unity.

(iii) decrease the age of retirement and increase consumption less than proportionately with wages if the utility function has a local inter-temporal elasticity of substitution less than unity.

Proof: Consider an increase in the whole time path of the wage schedule w_1 by a factor τ. We can then write the optimal retirement age and consumption level as function of the parameter τ, $R(\tau)$ and $c(\tau)$. The first order condition (17), which holds for all τ, is

$$u'(c(\tau))\tau w_1(R(\tau)) = \pi(R(\tau))$$

where

$$\pi(R(\tau)) = \frac{v(h(R(\tau), z_0)}{w_2(h(R(\tau), z_0)}$$

Clearly, the function π is increasing in R. Differentiating both sides of the first order condition we can derive

$$\frac{\tau R'(\tau)}{R(\tau)}\frac{R(\tau)\pi'(R(\tau))}{\pi(R(\tau))}-\frac{c(\tau)u''(c(\tau))}{u'(c(\tau))}\frac{\tau c'(\tau)}{c(\tau)}=1$$

or

$$\varepsilon_{R\tau}\varepsilon_{\pi R}+\frac{1}{\theta}\varepsilon_{c\tau}=1 \tag{36}$$

Where $\varepsilon_{R\tau}$ is the elasticity of the retirement age with respect to τ, $\varepsilon_{c\tau}$ is the elasticity of consumption with respect to τ, θ is the inter-temporal elasticity of substitution, and $\varepsilon_{\pi R}$ is the elasticity of π with respect to R, which we know to be positive. All these elasticities are local though we suppress their dependence on consumption and age for convenience.

From the budget constraint given by equation (9) we have that the effect of increasing wages by the proportion τ is

$$c(\tau)\int_0^T e^{-rt}s(t,z)dt \le \tau \int_0^{R(\tau)} e^{-rt}s(t,z)w_1(t)w_2(h(t,z))dt$$

It follows that changes to $c(\tau)/\tau$ and $R(\tau)$ have the same sign. Further, from this equation we can derive:

$$\begin{aligned}\varepsilon_{c\tau}=1 &\Leftrightarrow \varepsilon_{R\tau}=0\\ \varepsilon_{c\tau}>1 &\Leftrightarrow \varepsilon_{R\tau}>0\\ \varepsilon_{c\tau}<1 &\Leftrightarrow \varepsilon_{R\tau}<0\end{aligned}$$

Proposition 4 follows from these relationships and equation (36) .

Appendix 6

Gompertz-Makeham Survival Function: $s(\rho,a,b,t)=e^{\left\{-\rho t-\frac{a}{b}\left(e^{bt}-1\right)\right\}}$

Gompertz-Makeham Mortality Function: $m(\rho,a,b,t)=\rho+ae^{bt}$

Disutility of labour: $v(\rho, a, b, t) = d\left(\rho + ae^{bt}\right)$

Consumption growth path: $\frac{\dot{c}}{c} = (r - \partial)\frac{u'(c)}{-cu''(c)}$

Iso-elastic utility: $u(c) = \frac{c^{1-\beta}}{1-\beta}$ yields specific growth path: $\frac{\dot{c}}{c} = \frac{(r-\partial)}{\beta}$ with solution:

$$c(t) = c(0)e^{\frac{(r-\partial)}{\beta}t}$$

Implicit condition for retirement: $v(\rho, a, b, R) = u'(c)w(R)$ This, in our case with constant wage growth, is:

$$d\left(\rho + ae^{bR}\right) = w(0)e^{\sigma R}\left\{c(0)e^{\frac{(r-\partial)}{\beta}R}\right\}^{-\beta} = w(0)e^{(\sigma+\delta-r)R}\left\{c(0)\right\}^{-\beta} \tag{37}$$

We solve for $c(0)$ using the budget constraint:

$$\int_0^T e^{-rt}s(\rho, a, b, t)c(t)dt \leq \int_0^R e^{-rt}s(\rho, a, b, t)w(t)dt \tag{38}$$

The left hand side is:

$$c(0)\int_0^T e^{-\left(r+\rho-\frac{(r-\partial)}{\beta}\right)t-\frac{a}{b}\left(e^{bt}-1\right)}dt \tag{39}$$

By main result in Nordvall Lagerås (2010), equation (39) can be written as:

$$c(0)\left(\frac{a}{b}\right)^{\frac{\left(r+\rho-\frac{(r-\partial)}{\beta}\right)}{b}}\frac{e^{\frac{a}{b}}}{b}\Gamma\left\{\frac{-\left(r+\rho-\frac{(r-\partial)}{\beta}\right)}{b}, \frac{a}{b}\right\} \tag{40}$$

The right hand side of equation (38) can be written as:

$$\int_0^\infty e^{-rt}\mathrm{s}\left(\rho,a,b,t\right)w(t)dt-\int_R^\infty e^{-rt}\mathrm{s}\left(\rho,a,b,t\right)w(t)dt \tag{41}$$

The first part of equation (41) is:

$$\int_0^\infty e^{-rt}\mathrm{s}\left(\rho,a,b,t\right)w(t)dt=w\left(0\right)\int_0^\infty e^{-(\rho+r-\sigma)t-\frac{a}{b}\left(e^{bt}-1\right)}dt$$

And again, by Nordvall Lagerås's main result, this can be written as:

$$w\left(0\right)\left(\frac{a}{b}\right)^{\frac{(r+\rho-\sigma)}{b}}\frac{e^{\frac{a}{b}}}{b}\Gamma\left\{\frac{-\left(r+\rho-\sigma\right)}{b},\frac{a}{b}\right\} \tag{42}$$

By Nordvall Lagerås's corollary, the second part of equation (41) is written as:

$$w\left(0\right)\int_R^\infty e^{-(\rho+r-\sigma)t-\frac{a}{b}\left(e^{bt}-1\right)}dt=w\left(0\right)\int_0^\infty e^{-(\rho+r-\sigma)t-\frac{a}{b}e^{bt}\left(e^{bt}-1\right)}dt$$

And by Nordvall Lagerås's main result, the latter is:

$$w\left(0\right)\left(\frac{ae^{bR}}{b}\right)^{\frac{(r+\rho-\sigma)}{b}}\frac{e^{\frac{a}{b}e^{bR}}}{b}\Gamma\left\{\frac{-\left(r+\rho-\sigma\right)}{b},\frac{ae^{bR}}{b}\right\} \tag{43}$$

The difference between equations (42) and (43) gives us an expression for equation (41):

$$w\left(0\right)\left(\frac{a}{b}\right)^{\frac{(r+\rho-\sigma)}{b}}\frac{e^{\frac{a}{b}}}{b}\Gamma\left\{\frac{-\left(r+\rho-\sigma\right)}{b},\frac{a}{b}\right\}-w\left(0\right)\left(\frac{ae^{bR}}{b}\right)^{\frac{(r+\rho-\sigma)}{b}}\frac{e^{\frac{a}{b}e^{bR}}}{b}\Gamma\left\{\frac{-\left(r+\rho-\sigma\right)}{b},\frac{ae^{bR}}{b}\right\} \tag{44}$$

From equations (44), (40) and (38), we finally have a solution for

$c(0)$:

$$c(0)=\frac{w(0)\left[\left(\frac{a}{b}\right)^{\frac{(r+\rho-\sigma)}{b}}\frac{e^{\frac{a}{b}}}{b}\Gamma\left\{\frac{-(r+\rho-\sigma)}{b},\frac{a}{b}\right\}-\left(\frac{ae^{bR}}{b}\right)^{\frac{(r+\rho-\sigma)}{b}}\frac{e^{\frac{a}{b}e^{bR}}}{b}\Gamma\left\{\frac{-(r+\rho-\sigma)}{b},\frac{ae^{bR}}{b}\right\}\right]}{\left(\frac{a}{b}\right)^{\frac{\left(r+\rho-\frac{(r-\partial)}{\beta}\right)}{b}}\frac{e^{\frac{a}{b}}}{b}\Gamma\left\{\frac{-\left(r+\rho-\frac{(r-\partial)}{\beta}\right)}{b},\frac{a}{b}\right\}} \tag{45}$$

Substituting (45) into (37), we have our implicit function of the retirement age.

References

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