

# The Evolution of Population, Technology and Output

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## Abstract

This paper proposes a novel mechanism for the fertility decline that occurred across the world since the late 19th century. According to this, increases in the relative price of children and leisure goods due to technological progress induced parents to choose fewer children. To expose the role of this mechanism theoretically, it extends Galor and Weil's (2000) unified growth model by generalizing their utility function in which consumption and children (quantity augmented by quality) are unrelated, with a specification in which some commodities are unrelated with children while leisure goods are substitutes. Essentially, this gives rise to a fundamentally different reason for raising children from what are present in the literature, that is, the service provided by children is a leisure good from parents' point of view. Galor and Weil (2000) show that the fertility decline is caused by an increase in the rate of technological progress through a rise in educational attainment. In the modified model, in contrast, the fertility decline occurs naturally at high levels of income for a given level of education and a given price of leisure goods. Moreover, an increase in educational attainment and a decrease in the price of leisure goods speed up the transition from high to low fertility.

Key words: Fertility Transition, Technological Progress, Growth

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# 1 Introduction

The demographic transition from high to low fertility is a by-product of economic development that occurred across the world since the late 19th century.<sup>1</sup> There are various theories in the literature that try to explain the reasons behind this phenomenon - for example, the decrease in mortality (e.g., Kögel and Prskawets, 2001; Kalemlı-Ozcan, 2002, 2003; Lagerlof, 2003; Doepke, 2005; Tamura, 2006), the rise in income per capita (e.g., Becker, 1981; Jones, 2001), the old-age security hypothesis (e.g., Caldwell, 1976; Boldrin and Jones, 2002), the increase in educational attainment (e.g., Galor and Weil, 1999, 2000; Fernandez-Villaverde, 2001; Greenwood and Seshadri, 2002; Galor and Moav 2002; Doepke, 2004) and the decline in the gender gap (e.g., Galor and Weil, 1996).<sup>2</sup>

This paper gives rise to a fundamentally different reason for raising children from what are present in the literature and hence proposes a novel mechanism for the fertility decline. In the literature, children are treated as a durable consumption good that provides a stream of satisfaction (or service) to parents over their lifetime. Substantial amount of such satisfaction is produced when parents engage in activities such as playing with and talking to their children. Francis and Ramey (2008) consider these as high enjoyment activities and classify them as leisure. In that sense, some of the service provided by children can be considered as a leisure good and hence a reason for parents to have children in the economic analysis of fertility. Under such circumstances, one could think of other conventional leisure goods such as playing sports, going to cinemas, developing hobbies and socializing with others over drinks

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<sup>1</sup>The fertility transition began in developed countries towards the end of 19th century. The initial sharp decline in fertility was completed by World War II. After the baby boom, fertility has declined gradually ever since. In Asia and Latin America, the decline in population growth began towards the end of 20th century. In African countries, the demographic transition is about to begin due to decreased rates of fertility. For more evidence, see Galor (2005).

<sup>2</sup>See Galor (2005) for an extensive review of the literature.

that can be substituted for the service provided by children at certain degrees. Interestingly, consumption of such conventional leisure goods increases over the course of economic development. According to Lebergott (1996), the consumption share of leisure goods increased from 3 percent in 1900 to just over 8 percent in 2001. We attribute this fact to the fertility decline. The underlying mechanism is based on the observed increase in the relative price of children and conventional leisure goods. In the literature, the biggest cost of raising children is time so that the real wage is often used as the price of children which increases as economy develops. At the same time, the price of leisure goods decreases due to technological progress. According to Kopecky (2005), the relative price of leisure goods to CPI declined by 26 percent between 1900 and 1950. The idea is then when leisure goods were relatively more expensive in the early stages of economic development, parents had many children to gain utility from leisure activities during their lifetime as the opportunity cost of children in terms of foregone leisure goods was low. The increase in the relative price of children and leisure goods observed in the course of economic development may have induced parents to substitute leisure goods for children and hence fertility declined.

To expose the role of this mechanism theoretically, we extend Galor and Weil's (2000) (henceforth "GW") unified growth model. We choose GW's model for the following reason. Since Becker (1960), it has been a common practise in the economic analysis of fertility to consider that families derive utility from a single aggregate commodity (consumption) along with children (quantity augmented with or without quality) rather than the quantities of individual commodities. The reason for this is that, according to Becker (1960), there are no good or close substitutes for children. However, Becker admits that there may be many poor substitutes for children. GW assume a Cobb-Douglas utility function which implies no substitutes for children

(quantity augmented with quality) - i.e., the elasticity of substitution between children and consumption is unity. We disaggregate the consumption set in GW into two broad categories - consumption good and leisure good, and consider a more general utility function in which the former is unrelated with children while the latter is a substitute, at certain degrees.<sup>3</sup>

The rest of the model is the following. All individuals are assumed to produce both goods using different technologies. They use the same technology in GW to produce the consumption good which combines land and effective units of labor. The leisure good is, on the other hand, manufactured from the consumption good and subject to exogenous technological change. This simple technology allows us to capture the decrease in the price of leisure goods. Except for these modifications, the structure of the present model is the same as the GW model.

In this model, the decision rule for the optimal level of education (child quality) is the same as that in GW and depends positively on the rate of technological change expected to occur in the consumption good sector. The decision rule for the optimal number (quantity) of children, on the other hand, shows a hump-shaped relationship with the real wage, for a given level of education per child and a given price of the leisure good, rather than a positive (no) relationship for low (high) income levels in GW. The reason is that the price of the leisure good plays an important role when choosing child quantity as being substitutes for each other. Given a price of the leisure good, a rise in the real wage makes children relatively more expensive than

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<sup>3</sup>Considering substitutes for children is not new in the literature. Jones (2001) shows that fertility declines as the negative effect of rising wages (the opportunity cost of children) dominates its positive income effect. For the parameter value generating this outcome, his utility function also implies that births and consumption are substitutes. However, his focus is on the property of the utility function rather than identifying potential substitutes for children - i.e., the whole consumption set is a substitute for children. In contrast, we identify leisure goods as substitutes for the service provided by children from parents' point of view and hence emphasize the role of a falling price of leisure goods on the fertility decline.

the leisure good and hence fertility tends to decrease. At the same time, it generates a positive income effect and hence the number of children tends to increase. Initially, at low income levels, the positive effect dominates and hence fertility increases. At sufficiently high income levels, however, the negative effect dominates so that fertility declines.<sup>4</sup> Moreover, an increase in education per child (the mechanism of GW for the fertility decline) and a decrease in the price of leisure good due to technological progress speed up this fertility transition.

The remainder of the paper is organized as follows. Section 2 discusses the model and solves it. Section 3 analyzes the evolution of the dynamical system of the model and conclusions are given in Section 4.

## 2 The Model

We consider GW's overlapping-generations economy in which there are many identical individuals who live for two periods. As children in the first period of life, individuals are economically inactive and consume a fraction of their parents' time. As adults in the second period of life, they decide on the amount of consumption, the quantity (number) and quality (education) of their children and the labor market participation.

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<sup>4</sup>This result is quite similar to that generated by the mechanism based on the rise in income in Becker (1981) and Jones (2001). This theory, however, appears to be inconsistent with the demographic transition in the Western Europe countries which differed significantly in terms of income per capita. The feature that makes the current analysis different from these contributions is the decrease in the price of the leisure good which makes children more expensive. We do not find any studies compiling the price of leisure goods in Western Europe which would be useful to test the theory against its demographic transition directly. However, one could argue that these countries and the US experienced a similar drop in the price of leisure goods between 1900 and 1950 as being the most advanced economies. In that sense, the new mechanism contributed to the fertility decline occurred at least during the 20th century. Moreover, it is likely that the price of leisure goods declined in the 19th century. Although the Western European countries differed in income per capita, they experienced a similar growth rate of income per capita, indicating a similar rate of technological progress (e.g., Galor, 2005). GW and Galor and Moav (2002) argue that the acceleration in technological progress led to an increase in the rate of return to investments in human capital in these countries. One could make a similar argument that the acceleration in technological progress also led to a decrease in the price of leisure goods.

A consumption set in GW is now disaggregated into two types of goods: consumption good, an index of goods that are unrelated with children and leisure good, an index of leisure goods that, to some extent, are substitutes for children. All adults produce the two goods using different technologies. The consumption good is produced using land and efficiency units of labor as in GW in which the land is exogenous and fixed over time and the quantity of efficiency units of labor is endogenously determined from households' optimization problem in the previous period. The production of the leisure good, on the other hand, uses a fraction of the consumption good as an input. The demand for both goods together with the quantity and quality of children are determined by the households' decisions in each period.

## 2.1 Technology

Each adult produces  $z_t$  unit of the consumption good for each unit of time in period  $t$  in accordance with the following constant-returns-to-scale technology:

$$z_t = h_t^\alpha x_t^{1-\alpha} = h_t^\alpha \left( \frac{A_t X}{L_t} \right)^{1-\alpha} \quad (1)$$

where  $h_t$  is the amount of efficiency units of labor or human capital per adult,  $L_t$  is the size of the working age population,  $X$  is total (exogenous and constant) land,  $A_t$  is the level of land augmenting technology and  $\alpha \in (0, 1)$  is the labor income share. The term  $A_t X$  is total effective resources hence  $x_t = A_t X / L_t$  is effective resources per adult.<sup>5</sup>

All working age individuals manufacture the leisure good by combining a fraction of the consumption good and the current state of technology in this sub-sector.

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<sup>5</sup>Multiplying (1) by  $L_t$  yields the aggregate production function,  $Y_t = H_t^\alpha (A_t X)^{1-\alpha}$  where  $H_t$  is the aggregate amount of efficiency units of labor - i.e.,  $H_t = h_t L_t$ .

Specifically, we adopt a technology similar to that used by Vandenbroucke (2009) for his leisure good production and assume the following production function so as to simplify the analysis:

$$q_t = B_t m_t, \tag{2}$$

where  $q_t$  is the units of output of the leisure good,  $B_t$  is a productivity parameter, and  $m_t$  is the fraction of the consumption good. The maximization problem is given by:

$$\max_{m_t} \{p_t B_t m_t - m_t\}$$

where  $p_t$  is the unit price of the leisure good relative to that of the consumption good which is normalized to one. The optimal decision rule for this problem is  $p_t = 1/B_t$ .

## 2.2 Preferences and Budget Constraints

Each adult member of generation  $t$  is assumed to derive utility from the amount of the consumption good,  $c_t$ , in excess of its subsistence level  $\tilde{c} > 0$ , child quantity,  $n_t$ , augmented with human capital of each child,  $h_{t+1}$ , and the amount of the leisure good,  $d_t$ , in accordance with the following function:<sup>6</sup>

$$u = \gamma \log(c_t - \tilde{c}) + (1 - \gamma) \log [\mu (h_{t+1} n_t)^\sigma + (1 - \mu) d_t^\sigma]^{\frac{1}{\sigma}} \tag{3}$$

An important feature of the utility function in (3) is that the satisfaction from the consumption good is logarithmically separated from those generated from both children and the leisure good. The implication is that the consumption good is neither a

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<sup>6</sup>GW consider potential income of each child,  $w_{t+1} h_{t+1}$ , rather than human capital per child,  $h_{t+1}$ , in the parental utility function. Since they consider a Cobb-Douglas utility function which yields the same results as the log-separated function,  $w_{t+1}$  does not play any role as individuals take it as given. In the current model, however, considering  $w_{t+1} h_{t+1}$  would greatly complicate the analysis. For that reason, we follow Galor (2005) and consider only  $h_{t+1}$  in the utility function.

complement nor a substitute for both children and the leisure good - i.e., the elasticity of substitution between both  $c_t - \tilde{c}$  and  $h_{t+1}n_t$  and  $c_t - \tilde{c}$  as well as  $d_t$  is unity. Another, yet the most important feature of (3) is that the leisure good and children enter the utility function through a CES (Constant Elasticity of Substitution) function where  $\mu$  and  $1 - \mu$  denote their respective utility weights. The value of the parameter,  $\sigma$ , determines whether children and the leisure good are substitutes, complements or independent for parental utility. The elasticity of substitution is then  $1/(1 - \sigma)$ . When  $0 < \sigma \leq 1$  ( $\sigma < 0$ ), they are substitutes (complements), implying that the marginal utility of children decreases (increases) with an increase in the amount of the leisure good. If  $\sigma = 0$ , the expression converges to a Cobb-Douglas function, implying that the marginal utility of children is independent of the leisure good because the underlying utility function is logarithmic. The following analysis considers the case with  $0 < \sigma \leq 1$  to be consistent with the idea that the leisure good is a better substitute for children than the consumption good. Setting either  $\sigma = 0$  or  $\mu = 1$ , we can derive a log-separated utility function as in Galor (2005) which imply no substitutes for children in the economy as in GW.

As in GW, the adult is endowed with one unit of time which can be allocated between two mutually independent activities: working and raising children. Given her endowment of efficiency units of labor,  $h_t$ , the adult would earn potential income equal to  $z_t$  if she spent her entire time endowment in the labor market. Raising children requires only time as an input. Let  $\tau^q$  be the fraction of adult's time associated with producing and raising one child regardless of quality and  $\tau^e$  be the fraction of the adult's time connected with each level of education (quality) for each child,  $e_{t+1}$ . Since time devoted to raising children can be exchanged for the consumption good in the market, the opportunity cost of this activity is a fraction of her potential income:  $(\tau^q n_t + \tau^e e_{t+1} n_t) z_t$  which can also be understood as the total spending on

the purchase of children (both quality and quantity). Her actual income is then determined by  $(1 - \tau^q n_t - \tau^e e_{t+1} n_t) z_t$ . Since the adult allocates her actual income between purchasing the consumption and leisure goods, her budget constraint in GW is generalized to

$$c_t + p_t d_t = (1 - \tau^q n_t - \tau^e e_{t+1} n_t) z_t \quad (4)$$

where  $p_t$  is the relative price of the leisure good to that of the consumption good which is normalized to unity.

Unlike GW, we assume a constraint that governs the minimum number of children that parents want as in Jones (2001), that is,

$$n_t \geq \bar{n} > 0. \quad (5)$$

According to (5), the minimum number of children constraint binds if the optimal number of children derived from the optimization problem is smaller than it.

Following GW, we assume that human capital of each child,  $h_{t+1}$ , depends on the expected rate of technological progress between periods  $t$  and  $t + 1$  in the sector producing the consumption good,  $g_{t+1} \equiv (A_{t+1} - A_t)/A_t$  and education per child,  $e_{t+1}$  in an implicit fashion:

$$h_{t+1} \equiv h(g_{t+1}, e_{t+1}) \quad (6)$$

where  $h(\cdot) > 0$ ,  $h_g(\cdot) < 0$ ,  $h_{gg}(\cdot) > 0$ ,  $h_e(\cdot) > 0$ ,  $h_{ee}(\cdot) < 0$ , and  $h_{eg}(\cdot) > 0 \forall (e_{t+1}, g_{t+1}) \geq 0$ . The interpretation of these conditions is that the rate of technological progress has a negative effect on human capital and this "erosion effect" declines as  $g_{t+1}$  increases while education has a positive effect on human capital and its effect declines as  $e_{t+1}$  increases. The last property implies that technological progress increases the return to investments in education or education reduces the adverse effect

of technological progress.<sup>7</sup>

### 2.3 Utility Maximization

Given  $z_t$ ,  $p_t$  and  $g_{t+1}$ , the adults choose  $c_t, d_t, n_t$  and  $e_{t+1}$  to maximize their utility in (3) subject to the budget constraint in (4) and the minimum number of children constraint in (5). The first order conditions with respect to  $d_t, n_t$  and  $e_{t+1}$  are given by

$$\frac{(1-\gamma)p_t}{c_t - \tilde{c}} = \frac{\gamma(1-\mu)d_t^{\sigma-1}}{\mu(h_{t+1}n_t)^\sigma + (1-\mu)d_t^\sigma}, \quad (7)$$

$$\frac{(1-\gamma)(\tau^q + \tau^e e_{t+1})z_t}{c_t - \tilde{c}} = \frac{\gamma\mu h_{t+1}^\sigma n_t^{\sigma-1}}{\mu(h_{t+1}n_t)^\sigma + (1-\mu)d_t^\sigma}, \quad (8)$$

$$\frac{(1-\gamma)\tau^e n_t z_t}{c_t - \tilde{c}} = \frac{\gamma\mu h_{t+1}^{\sigma-1} n_t^\sigma h_e(\cdot)}{\mu(h_{t+1}n_t)^\sigma + (1-\mu)d_t^\sigma}. \quad (9)$$

In (7), the expression on the left-hand side represents the utility cost generated from purchasing one unit of the leisure good measured by the forgone consumption good while the utility gain is on the right-hand side. In (8), the expression on the right-hand side shows the utility gain of having one child while its utility cost is on the left-hand side as it decreases the amount of the consumption good by decreasing the adult's labor market participation. The price the adult pays for each child quantity is  $(\tau + \tau^e e_{t+1})z_t$  which is increasing in  $z_t$  as well as in  $e_{t+1}$  as the same level of education has to apply to each child. The expression in (9) shows the utility gain of purchasing an extra unit of education for each child on the right-hand side while the utility cost is on the left-hand side which is measured by the forgone consumption good through a decrease in the adult's time available for generating income. The price paid for the

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<sup>7</sup>Lagerlof (2006) considers an explicit functional form for  $h(g_{t+1}, e_{t+1})$ , that is  $h_{t+1} = \frac{e_{t+1} + \rho\tau}{e_{t+1} + \rho\tau + g_{t+1}}$  where  $\rho \in (0, 1)$  is an exogenous part of the total fixed time cost of raising children,  $\tau$ , that contributes towards building human capital - i.e., children acquire some knowledge while being raised (public good) which is, however, not as effective as formal education. Therefore  $e_{t+1} + \rho\tau$  is effective education.

extra unit of education is  $\tau^e n_t z_t$  which is an increasing function of  $z_t$  as well as of  $n_t$  because an additional unit of education must apply to more units. The presence of  $n_t$  in the price of quality and that of  $e_{t+1}$  in the price of quantity will ensure the classic interaction between the quality and quantity of children when the other things such as parental income change (e.g., Becker and Lewis, 1973).

We divide (8) by (9) and obtain the same expression as in GW:

$$G(e_{t+1}, g_{t+1}) \begin{cases} \leq 0 & \text{if } e_{t+1} = 0 \\ = 0 & \text{if } e_{t+1} > 0 \end{cases} \quad (10)$$

where  $G(e_{t+1}, g_{t+1}) = (\tau^q + \tau^e e_{t+1})h_e(g_{t+1}, e_{t+1}) - \tau^e h(e_{t+1}, g_{t+1})$ . Assuming  $G(0, 0) < 0$ , GW derive the following decision rule for the optimal level of education per child:

$$e_{t+1} = e(g_{t+1}) \begin{cases} = 0 & \text{if } g_{t+1} \leq \hat{g} \\ > 0 & \text{if } g_{t+1} > \hat{g} \end{cases} \quad (11)$$

where  $\hat{g} > 0$  and  $e'(g_{t+1}) > 0$  for any  $g_{t+1} > \hat{g}$ . According to (11), the optimal level of education per child is 0 when the rate of technological progress is sufficiently slow, but positive and increases with  $g_{t+1}$  for sufficiently fast technological progress. According to GW, a decrease in the level of human capital due to the erosion effect of technological progress is reduced by an increase in education for  $g_{t+1} > \hat{g}$ . The implication is that the overall effect of technological progress on human capital is still negative - i.e.,  $h_g(g_{t+1}, e(g_{t+1})) < 0$ .<sup>8</sup> This property is reflected in Lagerlof's (2006) choice of functional form for  $h(g_{t+1}, e_{t+1})$ . The optimal level of education per child hence their human capital are, however, independent of parental potential income,

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<sup>8</sup>This assumption simplifies the following analysis greatly. In the other cases, an increase in education would either fully recover or more than fully recover the loss in human capital - i.e.,  $h_g(e(g_{t+1}), g_{t+1}) \geq 0$ . While the former would not affect the qualitative results, the latter would lead to more general results.

$z_t$ . The reason can be explained using (8) and (9). Other things being equal, an increase in  $z_t$  tends to increase the demand for both  $e_{t+1}$  and  $n_t$  by decreasing the marginal utility of the consumption good,  $(1 - \gamma)/(c_t - \tilde{c})$ , implying that they are normal goods - i.e., it generates the wealth effect. At the same time, it produces a substitution effect on the demand for both commodities by increasing their prices  $\tau^e n_t z_t$  and  $(\tau^q + \tau^e e_{t+1}) z_t$ : directly as well as indirectly through  $n_t$  for the former and through  $e_{t+1}$  for the latter. This interaction between the quality and quantity of children leaves the level of education per child unaffected to the changes in  $z_t$ . In other words, the income and substitution effects are cancelled out.

After some manipulations, one may obtain the following expressions for the optimal quantities of children and the leisure good:

$$n_t = \max \left\{ \bar{n}, \frac{\gamma(1 - \tilde{c}/z_t)}{(\tau^q + \tau^e e_{t+1})(1 + \Omega_t)} \equiv n(p_t, g_{t+1}, z_t) \right\}, \quad (12)$$

$$d_t = M h_{t+1}^{\frac{\sigma}{\sigma-1}} \left[ \frac{z_t(\tau^q + \tau^e e_{t+1})}{p_t} \right]^{\frac{1}{1-\sigma}} n_t \equiv d(p_t, g_{t+1}, z_t) \quad (13)$$

where  $M = ((1 - \mu)/\mu)^{\frac{1}{1-\sigma}}$  and  $\Omega_t = M \left[ \frac{z_t(\tau^q + \tau^e e_{t+1})}{h_{t+1} p_t} \right]^{\frac{\sigma}{1-\sigma}}$ . Let  $n(\cdot)$  and  $d(\cdot)$  be shorthand notations for  $n(p_t, g_{t+1}, z_t)$  and  $d(p_t, g_{t+1}, z_t)$  respectively where we use  $e_{t+1} = e(g_{t+1})$  and  $h_{t+1} = h(e(g_{t+1}), g_{t+1}) = h(g_{t+1})$ . The functional properties of  $n(\cdot)$  and  $d(\cdot)$  are summarized in the following proposition.

**Proposition 1** *Other things being equal:*

(a) *a decrease in the price of the leisure good,  $p_t$ , leads to a decrease in  $n_t$ , but an increase in  $d_t$  - i.e.,*

$$n_p(\cdot) > 0 \quad \text{and} \quad d_p(\cdot) < 0,$$

(b) *technological progress expected to occur between time  $t$  and  $t + 1$  in the sector*

producing the consumption good,  $g_{t+1}$ , has a negative effect on  $n_t$  but an ambiguous effect on  $d_t$  for all  $g_{t+1}$  - i.e.,

$$n_g(\cdot) < 0 \quad \text{and} \quad d_g(\cdot) \begin{matrix} \leq \\ \geq \end{matrix} 0,$$

(c) there exists a time varying critical value  $\tilde{z}_t \equiv \tilde{z}(p_t, g_{t+1})$  so that an increase in the parental potential income,  $z_t$ , has (c.1) a positive effect on both  $n_t$  and  $d_t$  if  $z_t < \tilde{z}_t$  and (c.2) a negative effect on  $n_t$  but a positive effect on  $d_t$  if  $z_t > \tilde{z}_t$  - i.e.,

$$(c.1) \quad n_z(\cdot) \geq 0 \quad \text{and} \quad d_z(\cdot) > 0 \quad \text{if } z_t \leq \tilde{z}_t,$$

$$(c.2) \quad n_z(\cdot) \leq 0 \quad \text{and} \quad d_z(\cdot) > 0 \quad \text{if } z_t \geq \tilde{z}_t.$$

where  $\tilde{z}_t > \frac{\tilde{c}}{\sigma} > \tilde{c}$  for  $\sigma \in (0, 1)$ . In addition,  $\tilde{z}_g(\cdot) < 0$  and  $\tilde{z}_p(\cdot) > 0$ .

**Proof.** The results in parts (a) and (b) follow directly from differentiating  $n_t$  in (12) and  $d_t$  in (13) with respect to  $p_t$  and  $g_{t+1}$  after substituting (11) and (6) into (12) and (13). The sign of  $n_z(\cdot)$  in part (c) is determined by:

$$\text{sgn} \{n_z(\cdot)\} \equiv \text{sgn} \{\tilde{c}(1 - \sigma) - \Omega_t(\sigma z_t - \tilde{c})\}.$$

It is clear that  $n_z(\cdot) > 0$  for  $0 \leq z_t \leq \frac{\tilde{c}}{\sigma}$ , but  $n_z(\cdot) \begin{matrix} \geq \\ \leq \end{matrix} 0$  for  $z_t > \frac{\tilde{c}}{\sigma}$ . If  $\sigma = 0$ , it is true that  $n_z(\cdot) > 0$ . If  $\sigma = 1$ , the reverse is true. In the ideal case with  $\sigma \in (0, 1)$ , the second term,  $\Omega_t(\sigma z_t - \tilde{c})$ , is a monotonically increasing function of  $z_t$ . Since  $\tilde{c}(1 - \sigma)$  is constant, there exists a unique  $\tilde{z}_t \equiv \tilde{z}(p_t, g_{t+1})$  such that  $n_z(z_t = \tilde{z}_t) = 0$  or  $\tilde{c}(1 - \sigma) = \Omega_t(\sigma z_t - \tilde{c})$ . Given (11), (6) and  $h_g(g_{t+1}) < 0$ , the implicit function theorem suggests that  $\tilde{z}_g(\cdot) < 0$  and  $\tilde{z}_p(\cdot) > 0$ . ■

The intuitions of the results in Proposition 1 are the following. Define first the

relative price of child quantity to that of the leisure good,  $r_t$ , as  $r_t \equiv \frac{z_t(\tau^q + \tau^e e_{t+1})}{p_t}$ : an increase in both  $z_t$  and  $e_{t+1}$  but a decrease in  $p_t$  cause an increase in  $r_t$ . For  $g_{t+1} \leq \hat{g}$ , the relative price is  $r_t \equiv \frac{\tau^q z_t}{p_t}$ . The results in part (a) can be explained on the basis of the substitutability between children and the leisure good: the lower the price of the leisure good, the lower (higher) the demand for children (the leisure good). Since child quality ( $e_{t+1}$ ) does not depend on  $p_t$ , child quantity decreases but the leisure good increases directly due to an increase in  $r_t$ .

An increase in  $g_{t+1}$  has three negative effects on  $n_t$ . GW's quality-quantity tradeoff is one of them which makes  $n_t$  more expensive relative to  $e_{t+1}$  for  $g_{t+1} > \hat{g}$ . In general, even a small initial increase in  $e_{t+1}$  due to an increase in  $g_{t+1}$  can lead to a large decrease (increase) in  $n_t$  ( $e_{t+1}$ ) if the interaction between quality and quantity is strong (e.g., Becker and Lewis, 1973). The second negative effect makes child quantity more expensive relative to the leisure good by increasing  $r_t$  for  $g_{t+1} > \hat{g}$ . In that sense, the current setup strengthens the usual interaction between  $e_{t+1}$  and  $n_t$ . If there are no substitutes for children - i.e.,  $\sigma$  converges to zero, this effect will diminish to zero. The last effect of  $g_{t+1}$  on  $n_t$  arises through a decrease in  $h_{t+1}$ . As parents derive utility from the level of human capital of each child, a decrease in  $h_{t+1}$  decreases the marginal utility of child quantity according to (8) for  $\sigma \in (0, 1)$ . Again, if there are no substitutes for children ( $\sigma = 0$ ), this effect will vanish. The increase in  $g_{t+1}$ , on the other hand, has an ambiguous effect on  $d_t$ . A positive effect works through an increase in  $r_t$  and a decrease in  $h_{t+1}$  while a negative effect works through a decrease in  $n_t$ .<sup>9</sup>

An increase in  $z_t$  has opposite effects on  $n_t$ : the income (positive) and substitution (negative) effects. In a simple model with  $\tilde{c} = 0$  and  $\sigma = 0$ , these effects are exactly

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<sup>9</sup>For the functional forms used by Lagerlof (2006), we find that the positive effect dominates and the amount of the leisure good increases with  $g_{t+1}$ .

offset, leaving  $n_t$  unchanged. Introducing  $\tilde{c} > 0$  strengthens the income effect hence  $n_t$  increases. The strength of this additional income effect, however, becomes weaker as  $z_t$  increases. In the current setup with  $\sigma > 0$ , the substitution effect is also stronger than usual due to  $r_t > 0$ . At  $\tilde{z}_t$ , the sum of the usual and additional income effects is exactly offset by the stronger substitution effect. The case with  $z_t < \tilde{z}_t$  implies that the positive effects dominate the substitution effect hence the number of children increases with  $z_t$ . In other words, children are relatively cheaper than the leisure good in this region. In the case with  $z_t > \tilde{z}_t$ , however, children are more expensive than the leisure good (or the substitution effect dominates the income effect) hence child quantity decreases with  $z_t$ . Alternatively, the model generates a hump-shaped relation between  $n_t$  and  $z_t$  for given  $p_t$  and  $g_{t+1}$ . A decrease in  $p_t$  and an increase in  $g_{t+1}$  make the transition from  $n_z(\cdot) > 0$  to  $n_z(\cdot) < 0$  faster by decreasing  $\tilde{z}_t$ . The optimal amount of the leisure good increases with  $z_t$  for two reasons. Firstly, the leisure good is a normal good. Secondly, it is a substitute for children whose price increases with  $z_t$ , leading to an increase in  $r_t$ .

## 2.4 Technological Progress

The rate of technological progress that occurs between time  $t$  and time  $t + 1$  in the sector producing the consumption good,  $g_{t+1}$ , is the same as that in GW, that is, an implicit function of education at time  $t$ ,  $e_t$ , and the size of working age population at time  $t$ ,  $L_t$ :

$$g_{t+1} \equiv \frac{A_{t+1} - A_t}{A_t} = g(e_t, L_t) \quad (14)$$

where for  $L_t \gg 0$  and  $e_t \geq 0$ ,  $g(0, L_t) > 0$ ,  $g_i(\cdot) > 0$ , and  $g_{ii}(\cdot) < 0$ ,  $i = e_t, L_t$ . The rate of technological progress is an increasing concave function of each determinant for a sufficiently large population size. Moreover, there is positive technological

progress even if education is zero.<sup>10</sup> GW assume  $g_L(0, L_t) = 0$  for a sufficiently small population size to ensure that early stages of development take place in a Malthusian steady state. This assumption is kept in our analysis.

The rate of technological change in the production of the leisure good,  $g_{t+1}^B$ , is assumed to follow an exogenous process,

$$g_{t+1}^B \equiv \frac{B_{t+1} - B_t}{B_t} \geq 0, \quad (15)$$

so as to simplify the analysis. In the following analysis, the dynamical system of the economy is initially studied under the assumption that  $g_{t+1}^B = 0$  - i.e., the level of technology in this sector hence the price of the leisure good is constant. We will, however, analyze the effect of  $g_{t+1}^B > 0$  on the evolution of the dynamical system.<sup>11</sup>

## 2.5 Population, Technology and Effective Resources

The evolution of the size of working population,  $L_t$ , technology,  $A_t$ , and effective resources per worker,  $x_t$ , is the same as those in GW and is governed by the following three difference equations:

$$L_{t+1} = n_t L_t, \quad (16)$$

$$A_{t+1} = (1 + g_{t+1}) A_t, \quad (17)$$

$$x_{t+1} = \frac{1 + g_{t+1}}{n_t} x_t \quad (18)$$

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<sup>10</sup>Lagerlof (2006) considers the explicit form for  $g(e_t, L_t)$ , that is,  $g_{t+1} = (e_t + \rho\tau) \min\{\theta L_t, a^*\}$  where  $\theta > 0$  measures the "scale" effect of  $L_t$  while  $a^* > 0$  corresponds to  $\lim_{L \rightarrow \infty} g(\cdot) = a^*$  for given  $e_t$ . Thus population increases technological progress linearly for  $L_t \leq a^*/\theta$  and then has no effect.

<sup>11</sup>One could assume  $g_{t+1}^B = g_{t+1}$  which would not affect the results of the paper. This could, however, be a way to attribute the model to the demographic transition in Western Europe. The acceleration of technological progress during the last decades of the 19th century led to a decline in the price of leisure goods as well as an increase in the return to investment in education. Both would induce parents to have a fewer children.

where their initial levels are historically given at  $L_0$ ,  $A_0$  and  $x_0 = (A_0 X)/L_0$  respectively. The number of children per person,  $n_t$ , and the rate of technological progress,  $g_{t+1}$ , are determined by the expressions in (12) and (14) respectively.

Using (15), the evolution of the relative price of the leisure good,  $p_t = 1/B_t$ , can be written as

$$p_{t+1} = \frac{1}{1 + g_{t+1}^B} p_t \quad (19)$$

where  $p_0$  is historically given.  $g_{t+1}^B > 0$  implies that the leisure good becomes more affordable over time.

### 3 The Dynamical System

This section analyzes the dynamical system of the economy which determines its development through the evolution of population, income per capita, technology levels in the production of both the consumption and leisure goods, education per worker, human capital per worker and effective resources per worker. The sequence that determines the development of the economy in GW,  $\{e_t, g_t, x_t, L_t\}_{t=0}^{\infty}$ , is now extended to  $\{e_t, g_t, x_t, L_t, p_t\}_{t=0}^{\infty}$  in the current analysis that satisfies (14)-(19).

Since we do not follow GW in solving for the household's optimization problem, the dynamical system is characterized by one regime rather than two.<sup>12</sup> For a given

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<sup>12</sup>The utility maximization problem in GW is rather unconventional in the sense that they do not equate the marginal benefit and cost of children when choosing the optimal number of children. If they did, the decision rule for  $n_t$  would be  $n_t = \frac{\gamma(1-\epsilon/z_t)}{\tau^q + \tau^\epsilon e_{t+1}}$  according to which  $n_t$  is an increasing concave function of  $z_t$  and converges to  $\frac{\gamma}{\tau^q + \tau^\epsilon e_{t+1}}$  smoothly as  $z_t$  converges to infinity for given  $e_{t+1}$ . This rule can be derived as a special case of the one in the current model by setting either  $\mu = 1$  or  $p_t = \infty$  in (12) as  $\Omega_t = 0$ . The decision rule in GW is more or less an approximation to this rule in the sense that the same convergence is not smooth and takes place immediately when  $z_t = \tilde{z}$ . However, this particular approximation simplifies the dynamical system in GW greatly as the system is divided into two regions by  $\tilde{z}$ . Otherwise, the entire space  $(x_t, e_t)$ , in which the system is analyzed, would be the Malthusian region where  $z_t$  has always a positive effect on  $n_t$  as there is no  $\tilde{z}$ . As can be seen from Proposition 1, the current model generates  $\tilde{z}_t$  without following the optimization in GW.

size of population  $L$ , and a given price of the leisure good  $p$ , the development of the economy is determined by the following three-dimensional nonlinear system of difference equations:

$$\begin{cases} e_{t+1} = e(g(e_t); L) \\ g_{t+1} = g(e_t; L) \\ x_{t+1} = \phi(e_t, g_t, x_t; L, p)x_t \end{cases} \quad (20)$$

where  $\phi(e_t, g_t, x_t; L, p) \equiv (1 + g_{t+1})/n_t$  and the initial values  $e_0, g_0$  and  $x_0$  are historically given.

The evolution of  $e_t$  and  $g_t$  is independent of the  $x_t$ . Therefore, the analysis of the joint dynamics of education and technology is exactly the same as those in GW. Broadly speaking, this dynamical subsystem is characterized by three different configurations in the  $(e_t, g_t)$  space, depending on the size of population. For a small population size, there is a unique globally stable steady-state equilibrium  $(\bar{e}, \bar{g}) = (0, g^l)$  characterizing the dynamical subsystem (see Figure 3 in GW). For a moderate population size, the dynamical subsystem is characterized by multiple steady-state equilibria:  $(\bar{e}, \bar{g}) = (e^u, g^u)$  is unstable and lies between  $(\bar{e}, \bar{g}) = (0, g^l)$  and  $(\bar{e}, \bar{g}) = (e^h, g^h)$  which are stable. This is depicted in Figure 4 in GW. Figure 5 in GW shows the dynamical subsystem for a large population size which is characterized by a globally stable steady-state equilibrium  $(\bar{e}, \bar{g}) = (e^h, g^h)$ .

### 3.1 Global Dynamics

We analyze the evolution of the dynamical system of the economy in (20) using a series of phase diagrams in the  $(e_t, x_t)$  space, as described in GW. Each phase diagram, shown in Figures 2-4, has three components: the Malthusian Frontier which separates one regime where parental potential income has a positive effect on the chosen number

of children from the other where the effect is negative, the  $XX$  locus along which the effective resources per worker is constant and the  $EE$  locus along which the level of education per worker is constant. There is one similarity and two differences in the phase diagrams between ours and GW's. They are the same in terms of the  $EE$  locus. The first difference lies on the Malthusian Frontier which in GW separates the one regime where the correlation between parental income and child quantity is positive from the other where the correlation is zero. The second difference is on the shape of the  $XX$  locus because the current model generates a hump-shaped relationship between population growth and income.

**The Conditional Malthusian Frontier** According to Proposition 1, the economy switches the regime from the one where individuals' income has a positive effect on their chosen number of children to the other where the effect becomes negative when potential income  $z_t$  exceeds the time varying critical level  $\tilde{z}_t$ .

For the dynamical system in (20), the Conditional Malthusian Frontier,  $MM|_{g_t}$ , is the set of all pairs  $(e_t, x_t)$  conditional on given  $g_t$  and  $z_t = \tilde{z}_t \equiv \tilde{z}(g_{t+1}; p)$ . More formally,  $MM|_{g_t}$  is written as

$$MM|_{g_t} \equiv \{(e_t, x_t) : x_t^{1-\alpha} h(e_t, g_t)^\alpha = \tilde{z}(g_{t+1}) \mid g_t\}.$$

**Lemma 1** *If  $(e_t, x_t) \in MM|_{g_t}$ ,  $x_t$  is a monotonically decreasing function of  $e_t$ . Moreover, a decrease in  $x_t$  along  $MM|_{g_t}$  is larger than that in the case where  $\tilde{z}_t$  is constant for an equal increase in  $e_t$ . Furthermore, the critical  $\tilde{z}_t$  decreases along  $MM|_{g_t}$  as  $e_t$  increases.*

**Proof.** Given the result in part (c) of Proposition 1 that  $\tilde{z}_t$  is a decreasing function of  $g_{t+1}$ , an increase in  $e_t$  has a negative effect on  $\tilde{z}_t$  through an increase in  $g_{t+1}$ . Since

$h_t$  is an increasing function of  $e_t$ ,  $x_t$  must decrease in response to an increase in  $e_t$  along  $MM|_{g_t}$ . ■

The Conditional Malthusian Frontier is similar to that in GW in the sense that it is a downward sloping curve, intersects the  $x_t$  axis and asymptotically approaches to the  $e_t$  axis as  $x_t$  approaches to zero. As the functional forms are implicit, however, we cannot predict the second order property of the frontier while it is a strictly convex function in GW. Without loss of generality,  $MM|_{g_t}$  is depicted as a downward sloping convex curve in Figures 2-4.<sup>13</sup>

The frontier is affected by the evolution of population,  $L$ , and the price of the leisure good,  $p$ .

**Lemma 2** (a) *An increase in  $L$  and a decrease in  $p$  lead  $MM|_{g_t}$  to shift downward and leftward.*

**Proof.** According to (14), an increase in  $L$  leads to an increase in  $g_{t+1}$ . Given the result in part (c) of Proposition 1, both an increase  $g_{t+1}$  and a decrease in  $p$  have a negative effect on  $\tilde{z}_t$  so that  $x_t$  must decrease for given  $e_t$ . ■

The intuition of the results in Lemma 2 is that the boundary of the Malthusian region where potential income has a positive effect on child quantity will shrink as the leisure good becomes cheaper and population size increases.<sup>14</sup>

**The  $XX$  Locus** According to (18), the effective resources per worker,  $x_t$ , is constant if growth rates of working population and technology are equal. The conditional  $XX$

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<sup>13</sup>The second order property of the Conditional Malthusian Frontier would be irrelevant to the qualitative analysis.

<sup>14</sup>By setting  $p_t = \infty$  and following GW's utility maximization approach, we can derive their frontier as  $\Omega_t = 0$  in (12).

locus is the set of all pairs  $(e_t, x_t)$  for given  $g_t$ , such that  $x_t$  is in a steady state. More formally,

$$XX|_{g_t} \equiv \{(e_t, x_t) : x_{t+1} = x_t \mid g_t\}.$$

**Lemma 3** *There exists a unique value  $0 < \hat{e} < e^h$  such that (a) for each  $0 \leq e_t < \hat{e}$ , there are two values for  $x_t \in XX|_{g_t}$ :  $x_t^h > x_t^l$  such that  $z(e_t, x_t^h) > \tilde{z}_t$  and  $z(e_t, x_t^l) < \tilde{z}_t$ , and a unique value  $x_t^l < \tilde{x}_t < x_t^h$  such that  $z(e_t, \tilde{x}_t) = \tilde{z}_t$ ; (b) for  $e_t = \hat{e}$ , there is a unique  $\hat{x} \in XX|_{g_t}$  such that  $z(\hat{e}, \hat{x}) = \tilde{z}_t$ ; and (c) for  $\hat{e} < e_t \leq e^h$ , there is no  $x_t \in XX|_{g_t}$ . Moreover, for  $z_t \geq \tilde{z}_t$ ,*

$$x_{t+1} - x_t \begin{cases} > 0 & \text{if } [(e_t, x_t) > (e_t, x_t^h(e_t)) \text{ and } 0 \leq e_t < \hat{e}], [(\hat{e}, x_t) > (\hat{e}, \hat{x})] \text{ or } [e_t > \hat{e}] \\ = 0 & \text{if } [(e_t, x_t) = (e_t, x_t^h(e_t)) \text{ and } 0 \leq e_t < \hat{e}] \text{ or } [(e_t, x_t) = (\hat{e}, \hat{x})] \\ < 0 & \text{if } (e_t, x_t) < (e_t, x_t^h) \text{ and } 0 \leq e_t < \hat{e}. \end{cases}$$

For  $z_t \leq \tilde{z}_t$ ,

$$x_{t+1} - x_t \begin{cases} < 0 & \text{if } (e_t, x_t) > (e_t, x_t^l(e_t)) \text{ and } 0 \leq e_t < \hat{e} \\ = 0 & \text{if } [(e_t, x_t) = (e_t, x_t^l(e_t)) \text{ and } 0 \leq e_t < \hat{e}] \text{ or } [(e_t, x_t) = (\hat{e}, \hat{x})] \\ > 0 & \text{if } [(e_t, x_t) < (e_t, x_t^l(e_t)) \text{ and } 0 \leq e_t < \hat{e}], [(\hat{e}, x_t) < (\hat{e}, \hat{x})] \text{ or } [e_t > \hat{e}]. \end{cases}$$

**Proof.** Rewrite (12) as follows:

$$n_t = \frac{\gamma(1 - \tilde{c}/z_t)}{(\tau^q + \tau^e e_{t+1}) \left(1 + \Psi_t z_t^{\frac{\sigma}{1-\sigma}}\right)} \equiv n(e_t, x_t) \quad (21)$$

where  $\Omega_t = \Psi_t z_t^{\frac{\sigma}{1-\sigma}}$ ,  $\Psi_t = M \left(\frac{\tau^q + \tau^e e_{t+1}}{h_{t+1} p}\right)^{\frac{\sigma}{1-\sigma}}$ ,  $e_{t+1} = e(g_{t+1})$ ,  $h_{t+1} = h(g_{t+1})$ ,  $g_{t+1} = g(e_t; L)$  and  $z_t = x_t^{1-\alpha} h(e_t, g_t)^\alpha$ . We find that  $n_x(e_t, x_t) = \tilde{c}(1 - \sigma) - F(x_t, e_t)$  where  $F(x_t, e_t) = \Psi_t z_t^{\frac{\sigma}{1-\sigma}} (\sigma z_t - \tilde{c})$  which is monotonically increasing in  $x_t$  for each  $e_t$  - i.e.,

$F_x(x_t, e_t) > 0$ .<sup>15</sup> Since  $\tilde{c}(1 - \sigma)$  is independent of  $x_t$ , there exists a unique  $\tilde{x}_t$  for given  $e_t$ , such that  $\tilde{c}(1 - \sigma) = F(x_t, e_t)$  and  $\tilde{z}_t = \tilde{x}_t^{1-\alpha} h(e_t, g_t)^\alpha$  - i.e., the pair  $(\tilde{x}_t, e_t)$  is on  $MM|_{g_t}$ . Both  $\tilde{c}(1 - \sigma)$  and  $F(x_t, e_t)$  are depicted in panel (a) of Figure 1 where  $\ddot{x}(e_t) = (\tilde{c}/(\sigma h_t^\alpha))^{1/(1-\alpha)}$  such that  $\sigma z_t = \tilde{c}$ .<sup>16</sup> For  $x_t < \tilde{x}_t$ ,  $\tilde{c}(1 - \sigma) > F(x_t, e_t)$  hence  $n_x(x_t, e_t) > 0$ . For  $x_t > \tilde{x}_t$ ,  $\tilde{c}(1 - \sigma) < F(x_t, e_t)$  hence  $n_x(x_t, e_t) < 0$ . Thus there is a hump-shaped relationship between  $n_t$  and  $x_t$  for each  $e_t$ . Hence we can obtain  $\tilde{n}_t = n(\tilde{x}_t, e_t)$  such that  $n_x(\tilde{x}_t, e_t) = 0$ . Now depict both  $1 + g(e_t)$  and  $n(x_t, e_t)$  in the  $(n_t, x_t)$  space which is given in panel (b) of Figure 1 where  $\bar{x}(e_t)$  can be determined from (12) such that  $n_t = \bar{n}$ . Since  $1 + g(e_t)$  is independent of  $x_t$ , it is a horizontal line. The intersection of  $1 + g(e_t)$  and  $n(x_t, e_t)$  yields two steady-state values,  $x^l(e_t)$  and  $x^h(e_t)$ , such that  $x^l(e_t) < \tilde{x}(e_t) < x^h(e_t)$  if  $\tilde{n}_t > 1 + g(e_t)$ . Furthermore,  $x_{t+1} - x_t > 0$  for both  $x_t < x^l(e_t)$  and  $x_t > x^h(e_t)$  as  $n_t < 1 + g(e_t) < \tilde{n}_t$ . However,  $x_{t+1} - x_t < 0$  for  $x^l(e_t) < x_t < x^h(e_t)$  as  $1 + g(e_t) < n_t \leq \tilde{n}_t$ .

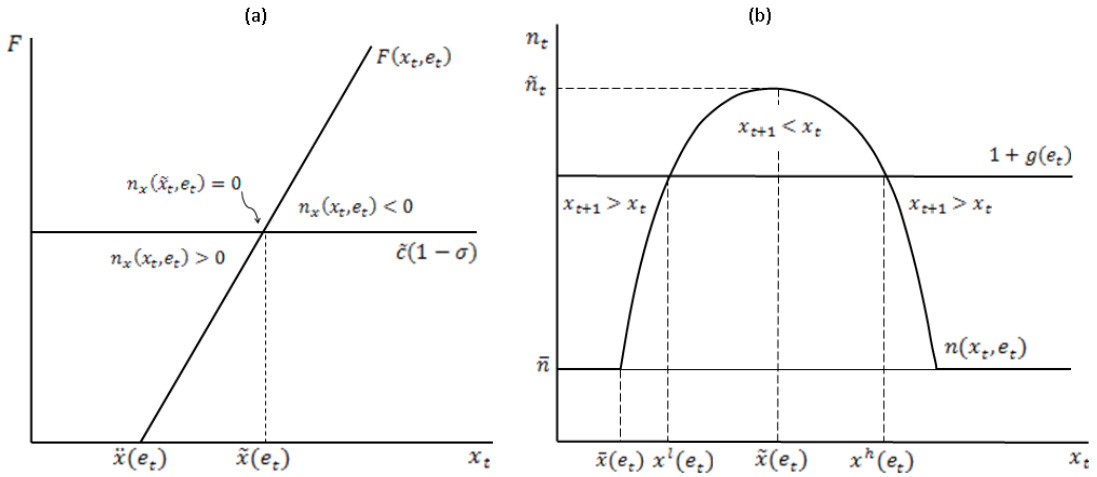


Figure 1. Derivation of the  $XX$  locus

<sup>15</sup>This is essentially the same expression found to prove the result in part (c) of Proposition 1.

<sup>16</sup>The second order property of  $F(x_t, e_t)$  is ambiguous depending the values of  $\alpha$  and  $\sigma$ . However, the qualitative analysis is not affected by this. Hence without loss of generality,  $F(x_t, e_t)$  is depicted as a straight line in Figure 1.

Let us now analyze the situation where  $e_t$  increases. For each  $x_t$ ,  $F(x_t, e_t)$  increases as both  $z_t$  and  $\Psi_t$  increase. For the new  $e_t$  and  $\Psi_t$ , a decrease in  $\tilde{x}_t$  must be sufficient to restore  $\tilde{c}(1 - \sigma) = \Psi_t \tilde{z}_t^{\frac{\sigma}{1-\sigma}} (\sigma \tilde{z}_t - \tilde{c})$ . In particular, the new  $\tilde{x}_t$  must be smaller than before to ensure that the new  $\tilde{z}_t$  is smaller than before. Thus  $\tilde{n}_t$  is smaller than before. It implies that the point,  $(\tilde{n}_t, \tilde{x}_t)$ , shifts leftward and downward in the  $(n_t, x_t)$  space. In other words,  $\tilde{n}_t$  decreases monotonically as  $e_t$  increases. Since  $1 + g(e_t)$  increases as  $e_t$  increases, there exists a unique  $\hat{e}$  such that  $\tilde{n}_t = 1 + g(\hat{e})$ . For  $\tilde{n}_t = 1 + g(\hat{e})$ ,  $\tilde{x}_t = \hat{x}$  and the pair  $(\hat{e}, \hat{x})$  is on  $MM_{|g_t}$ . Since  $\tilde{n}_t < 1 + g(e_t)$  for  $\hat{e} < e_t \leq e^h$ , there is no  $x_t \in XX_{|g_t}$ . ■

The locus  $XX_{|g_t}$  is strictly below the curve  $MM_{|g_t}$  for  $e_t < \hat{e}$  and  $x_t < \hat{x}$ , but strictly above the curve  $MM_{|g_t}$  for  $e_t < \hat{e}$  and  $x_t > \hat{x}$ . At  $(\hat{e}, \hat{x})$ , the curve  $MM_{|g_t}$  and the locus  $XX_{|g_t}$  coincide. Since the curve  $n(x_t, e_t)$  shifts leftward and downward as  $e_t$  increases,  $x_t^l$  may either increase or decrease as in GW while  $x_t^h$  decreases unambiguously. Hence without loss of generality, the part of the locus  $XX_{|g_t}$  below the curve  $MM_{|g_t}$  is depicted in Figures 2-4 as an upward sloping curve as in GW.<sup>17</sup>

The locus  $XX_{|g_t}$  is also affected by the evolution of population and the price of the leisure good.

**Lemma 4** *An increase in  $L$  and a decrease in  $p$  lead  $XX_{|g_t}$  under  $MM_{|g_t}$  to shift up, but above  $MM_{|g_t}$  to shift down. Furthermore, the critical  $\hat{e}$  decreases as  $L$  increases and  $p$  decreases.*

**Proof.** An increase in  $L$  leads to  $\phi(\cdot) = (1 + g_{t+1})/n_t > 1$  through its direct positive effect on  $g_{t+1}$  and an indirect negative effect on  $n_t$  which works through an increase in  $e_{t+1}$ , a decrease  $h_{t+1}$  and hence an increase in  $\Psi_t$ . Thus  $n_t$  must increase to restore

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<sup>17</sup>GW assume that the total derivative  $\partial z(h_t, x_t)/\partial g_t > 0$  (holding  $A_{t-1}$  constant) although the partial derivative is negative (holding  $x_t$  and thus  $A_t$  constant). This assumption is maintained in the current model.

the steady-state condition for  $x_t$ ,  $\phi(\cdot) = 1$ . It must be achieved as  $x_t^l$  increases (i.e.,  $z_t$  increases) and  $x_t^h$  decreases (i.e.,  $z_t$  decreases) for each  $e_t$ . The increase in  $\Psi_t$  leads to a decrease in the corresponding  $\tilde{x}_t$  by shifting  $F(x_t, e_t)$  leftward which is consistent with the result in Lemma 2. For given  $e_t$ , a decrease in  $\tilde{x}_t$  leads to a decrease in  $\tilde{z}_t$  so that  $\tilde{n}_t$  decreases. Thus  $\hat{e}$  decreases. A decrease in  $p$  has the same effect which works through a decrease in  $n_t$  for  $\phi(\cdot)$ . ■

An interesting intuition of the result in Lemma 4 is that the economy may cross over the Malthusian Frontier into the Modern Growth region as the price of the leisure good decreases for a given level of education.<sup>18</sup>

**The  $EE$  locus** The conditional  $EE$  locus is exactly the same as that in GW, that is a set all pairs  $(e_t, x_t)$  conditional on given  $g_t$  such that education per worker  $e_t$  is in a steady state:

$$EE|_{g_t} \equiv \{(e_t, x_t) : e_{t+1} = e_t \mid g_t\}.$$

GW shows that the steady-state values of  $e_t$  are independent of  $g_t$  and  $x_t$ , for a given size of population. Therefore the locus  $EE$  is a vertical line in the  $(e_t, x_t)$  space and shifts rightward as population size increases. The location of the locus  $EE$  identifies one of three phases of economic development in terms of the evolution of education and technology. In the early stage of development, the locus  $EE$  is vertical at  $e = 0$  representing the globally stable temporary steady-state equilibrium,  $(\bar{e}, \bar{g}) = (0, g^l)$  and

$$e_{t+1} - e_t \begin{cases} = 0 & \text{if } e_t = 0 \\ < 0 & \text{if } e_t > 0. \end{cases} \quad (22)$$

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<sup>18</sup>The locus  $XX|_{g_t}$  in GW is vertical at  $\hat{e}$  above the Malthusian Frontier which can be derived by setting  $p_t = \infty$  as well as following their utility maximization approach. Under such circumstances, one must maintain the assumption (A4) in GW to ensure that the  $XX$  locus is nonempty for  $z_t \geq \tilde{z}$ , that is,  $\hat{g} < (\gamma/\tau^q) - 1 < g(e^h(L_0), L_0)$ .

The intermediate stage of development is characterized by the multiple locally stable temporary steady-state equilibria,  $(0, g^l)$ ,  $(e^u, g^u)$  and  $(e^h, g^h)$  so that the locus  $EE$  is vertical at  $e_t = 0$ ,  $e_t = e^u$  and  $e_t = e^h$ . The ones at  $e_t = e^u$  and  $e_t = e^h$  shift rightward as population size increases. The global dynamics of  $e_t$  are given by

$$e_{t+1} - e_t \begin{cases} = 0 & \text{if } e_t \in \{0, e^u, e^h\} \\ > 0 & \text{if } e^u < e_t < e^h \\ < 0 & \text{if } 0 < e_t < e^u \text{ or } e_t > e^h. \end{cases} \quad (23)$$

In the advanced stage of development, the locus  $EE$  at  $e_t = e^h$  represents a globally stable steady-state equilibrium,  $(e^h, g^h)$ . It shifts rightward as population size increases. The global dynamics of  $e_t$  in this case is given by

$$e_{t+1} - e_t \begin{cases} = 0 & \text{if } e_t = e^h \\ > 0 & \text{if } 0 \leq e_t < e^h \\ < 0 & \text{if } e_t > e^h. \end{cases} \quad (24)$$

### 3.2 Conditional Steady-State Equilibria

The dynamical system in the early stage of economic development with small population sizes is characterized by two conditional steady-state equilibria which are given by the intersection between the  $XX$  locus and the  $EE$  locus in the  $(e_t, x_t)$  space, as shown in Figure 2.

Both equilibria are conditional on the rate of technological progress, the size of population and the price of the leisure good. Since the conditional steady-state equilibrium  $(\bar{e}, \bar{x}) = (0, x^h)$  is unstable, the locally stable conditional steady-state equilibrium  $(\bar{e}, \bar{x}) = (0, x^l)$  is the Malthusian steady-state. Another reason why the unstable steady-state equilibrium is not the Malthusian steady-state is that an in-

crease in potential income has a negative effect on child quantity which is opposite to the assumption on which the Malthusian model is built.

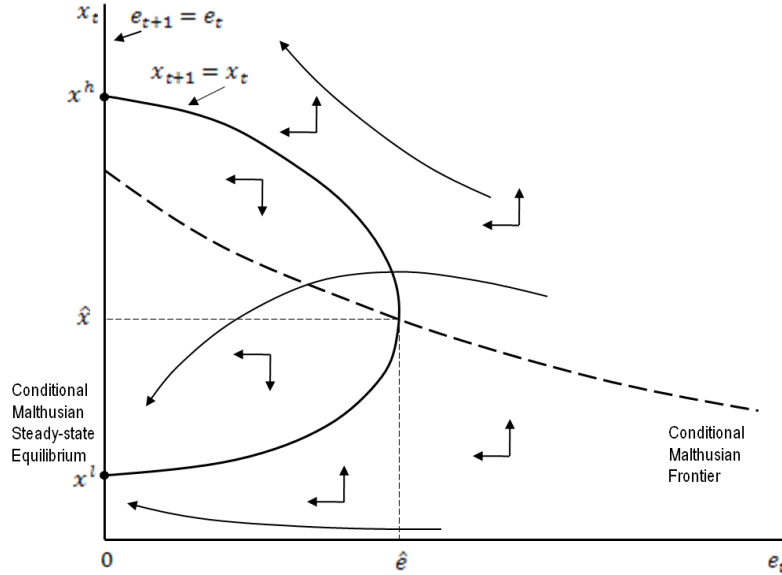


Figure 2. The Conditional Dynamical System in the Early Stage

The dynamical system in the intermediate stage of development with moderate population sizes, depicted in Figure 3, is similar to that in GW in the sense that the Malthusian conditional steady-state is locally stable and the conditional steady-state equilibrium  $(e^u, x^l(e^u))$  is a saddlepoint. In addition to those in GW, there are two conditional unstable steady-state equilibria:  $(0, x^h(0))$  and  $(e^u, x^h(e^u))$ . If the level of education is above  $e^u$ , the dynamical system converges to an equilibrium with a level of education  $e^h$  and possibly a steady-state growth rate of  $x_t$ , given the population size and the price of the leisure good. In the advanced stage of development with large population sizes, the dynamical system is, as depicted in Figure 4, converges globally to an education level  $e^h$  and possibly a steady-state growth rate of  $x_t$ , given the population size and the price of the leisure good.

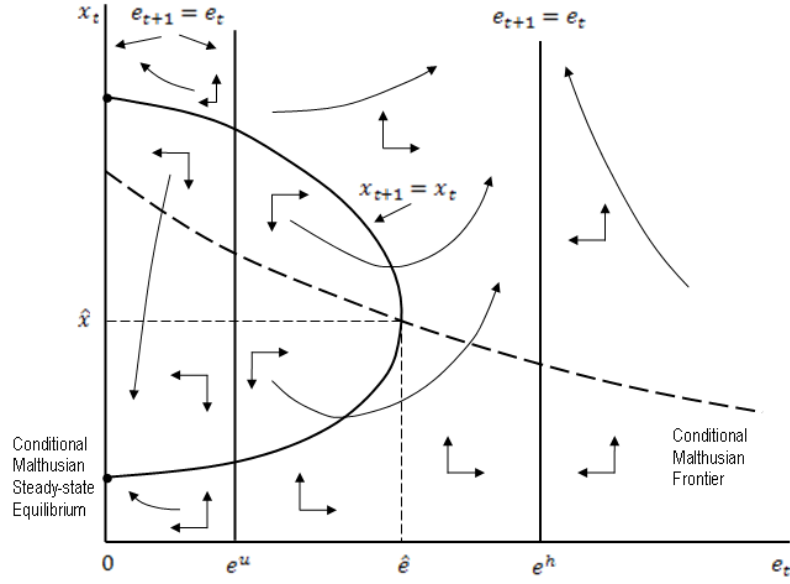


Figure 3. The Conditional Dynamical System in the Intermediate Stage

### 3.3 Analysis

Collecting the results obtained so far, we can now discuss the process of economic development from a Malthusian regime to a Modern Growth regime through a Post-Malthusian regime and a demographic transition. Consider that the economy is in the early stage of development where population size is sufficiently small, the rate of technological progress is so slow that parents find it inefficient to invest in their children's education. In addition, the price of the leisure good is extremely high as the technology of producing these goods is so primitive that its consumption/production is small and children are very cheap relative to the leisure good. In Figure 2, the situation is represented by the temporary, conditional and locally stable Malthusian steady-state equilibrium where both the level of education and effective resources per worker are constant, for a given rate of technological change. Consequently, output

per capita is constant from (1) and (11). Thus population is growing slowly, at the same rate as technological progress. For an extremely high price of the leisure good, the unstable conditional steady-state equilibrium,  $(0, x^h)$ , may not exist so that the Malthusian steady-state equilibrium can be globally stable as the  $XX$  locus for  $z_t > \tilde{z}_t$  (i.e., above the Malthusian Frontier) may become vertical at  $\hat{e}$  as in GW. Temporary shocks to technology and population will be adjusted towards the Malthusian steady-state.

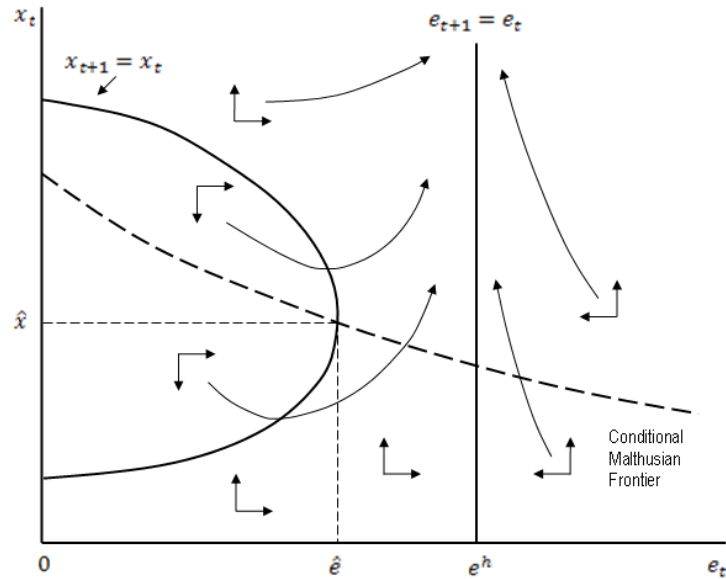


Figure 4. The Conditional Dynamical System in the Advanced Stage

Over time, the price of the leisure good may decrease and create the unstable conditional steady-state equilibrium (i.e., the  $XX$  locus for  $z_t > \tilde{z}_t$  is no longer vertical at  $\hat{e}$ ). If the effective resources per worker jump above the  $XX$  locus for  $z_t > \tilde{z}_t$  due to temporary shocks to population and technology, the system will converge to zero education level and possibly a steady-state growth rate of  $x_t$ , given the population size, the rate of technological progress and the price of the leisure good. Under such

circumstances, the number of children per person will decrease until the minimum quantity of children constraint binds as children are expensive relative to the leisure good in this region. In other words, it will generate a demographic transition. However, we do not consider this possibility. Instead, we assume that the economy stays around the Malthusian steady-state at this early stage of development.

When the size of the population reaches a sufficiently high level, the dynamical system will be characterized by multiple steady-state equilibria as in GW: the Malthusian steady-state with constant income per capita, slow technological progress and zero education, and the Modern Growth steady-state with fast technological progress, a high level of education and increasing income per capita. As depicted in Figure 3, the convergence towards these steady states is history dependent. In addition, the economy may jump over the boundary and evolve accordingly due to shocks to technology and education. Like GW, however, we are interested in the economy starting out in the Malthusian steady-state so that it stays there at this intermediate stage.

Figure 5 in GW shows that the evolution of education and technology is monotonic and converges to a unique globally stable steady-state with fast technological progress and a high level of education when the size of population reaches a sufficiently high level. Simultaneously, the Malthusian steady-state disappears. Exactly the same mechanism is at work in the current analysis. As in GW, technological progress has opposite effects on the evolution of population growth in the Malthusian region of Figure 4. As shown in Proposition 1, technological progress has a negative effect on the number of children per adult through an increase in education per child and an overall decrease in human capital per child, for a given level of potential income and a given price of the leisure good. At the same time, it increases parental potential income which has a positive effect on the number of children per person. If the price of the leisure good decreases, child quantity will tend to decrease. Initially, the

positive effect dominates all the negative effects hence the rate of population growth will increase, reflecting the characteristics of the Post-Malthusian regime.

If the positive effect of technological change continues to dominate the negative effects on the number of children per person, the rate of population growth will increase continuously until the economy crosses over the Malthusian Frontier. As soon as the economy enters the Modern Growth region, the rate of population growth will decrease unambiguously as the growth in parental potential income produces a negative effect on the number of children per person. Income per capita continues to increase while the rate of population growth continues to decrease even without any further improvements in technology (i.e., both education per person and human capital per person are constant). The source of growth for income per capita will be the growth of effective resources per capita as the rate of technological change is faster than population growth in this region. This is one of the main differences between the current analysis and that in GW. Under the same circumstances, GW would predict a constant growth rate of population but growing income per capita as parental potential income has no effect on fertility. Only faster technological changes lead to a decrease in population growth by raising education. In the current model, in contrast, an increase in education lead to an even faster decline in population growth as it complements with the negative effect of growing income per capita.

Let us now discuss the role of decreasing price of the leisure good on the dynamics of population and output. A decrease in the price of the leisure good speeds up the transition into the Modern Growth region by shifting the Malthusian Frontier down (i.e., decreasing the critical level of education). To emphasize it, suppose that the highest level of education falls below its critical level and stays unchanged - i.e.,  $e_t = e^h < \hat{e}$ . If it was GW's economy, it would stay in the globally stable steady-state

characterized by the intersection of the  $XX$  and  $EE$  locuses.<sup>19</sup> In the current model, however, a continuous decrease in the price of the leisure good reduces the critical level of education below the existing level of education in accordance with Lemma 4. Consequently, the economy moves into the Modern Growth regime. In other words, it is possible for an economy to experience the characteristics of the Modern Growth regime for a lower level of education as children become more expensive relative to the leisure good. In that sense, the current model offers an alternative mechanism for the transition out of the Malthusian stagnation.

The decrease in population growth is bounded from below as the minimum quantity of children constraint binds. If the minimum quantity of children is one per adult, population growth will be zero in the Modern Growth regime. Under such circumstances, the economy converges to a global steady state in which both education and the rate of technological progress are constant as population size is constant. If the minimum number of children above (below) one, the size of population will increase (decrease) and hence the evolution of education and technology will move accordingly.

## 4 Conclusion

The paper has discussed the role of the rise in the relative price of children and leisure goods over time in explaining the fertility decline observed in the most of the countries in the world in the course of economic development. To expose this mechanism, it has extended the unified growth model of GW by generalizing their utility function in which consumption and children are independent of each other,

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<sup>19</sup>This is possible for a sufficiently large population size. According to Jones (1995), growth models should imply a lack of scale effects. Interestingly, such models are directed towards the modern regime. Lagerlof (2006) limits the scale effect of population by considering  $g_{t+1} = (e_t + \rho\tau) \min\{\theta L_t, a^*\}$  where  $\theta > 0$  and  $a^*$  is constant. For  $L_t > a^*/\theta$ , there would be a unique steady state with  $g_{t+1} = g^h$  and  $e_{t+1} = e^h$ .

with one in which some commodities independent of children while leisure goods are substitutes for children. Consequently, the demographic transition from high to low fertility is the outcome of three events in this model, rather than one in GW. For a given level of education and a given price of leisure goods, the fertility decline is a natural phenomenon when children becomes relatively more expensive than leisure goods in enhancing parental welfare, at high income levels. An increase in educational attainment (which is the mechanism in GW) and a decrease in the price of leisure goods lead to a faster fertility transition by making child quantity even more expensive from parents' point of view.

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